1. Introduction

Let \((M, g)\) be a \((3+1)\)-dimensional vacuum globally hyperbolic space-time, i.e. \(g\) is a Lorentz metric of signature \((-+,+,+,+)\) satisfying the Einstein vacuum equations

\[ \text{Ric}(g) = 0 \]

and every causal curve intersects a Cauchy surface at precisely one point. If \((M, g)\) has a compact, constant mean curvature (CMC) Cauchy surface \(\Sigma_0\) with mean curvature \(t_0 < 0\), then there exists a foliation of a neighborhood of \(\Sigma_0\) by compact CMC surfaces, and the mean curvature varies monotonically from slice to slice. The CMC conjecture states that there is a foliation in \(M\) of CMC Cauchy surfaces with mean curvatures taking on all allowable values, i.e. the mean curvatures take all values in \((-\infty, 0)\) if \(\Sigma_0\) is of Yamabe type \(-1\) or 0, while the mean curvatures take on all values in \((-\infty, \infty)\) if \(\Sigma_0\) is of Yamabe type \(+1\). Certain progress has been made \([3]\), the CMC conjecture however remains open. One of the important steps to attack the CMC conjecture is to provide a reasonable breakdown criterion to detect what may happen when the CMC foliation can not be extended.

In order to set up the framework, in this paper we assume that \(M_*\) is a part of the space-time \((M, g)\) foliated by CMC hypersurfaces \(\Sigma_t\) with mean curvature \(t\) satisfying \(t_0 \leq t < t_*\) for some \(t_0 < t_* < 0\). We shall refer to \(\Sigma_0 := \Sigma_{t_0}\) as the initial slice. Thus, \(M_* = \bigcup_{t \in [t_0, t_*)} \Sigma_t\) with \(t_* < 0\) and there is a time function \(t\) defined on \(M_*\), monotonically increasing toward the future, such that each \(\Sigma_t\) is a level hypersurface of \(t\) with the lapse function \(n\) and the second fundamental form \(k\) defined by

\[ n := (-g(Dt, Dt))^{1/2} \quad \text{and} \quad k(X, Y) := -g(D_X T, Y), \]

where \(T\) denotes the future directed unit normal to \(\Sigma_t\), \(D\) denotes the space-time covariant differentiation associated with \(g\), and \(X, Y\) are vector fields tangent to \(\Sigma_t\). Let \(g\) be the induced Riemannian metric on \(\Sigma_t\) and let \(\nabla\) be the corresponding covariant differentiation. For any coordinate chart \(\mathcal{O} \subset \Sigma_0\) with coordinates \(x = (x^1, x^2, x^3)\), let \(x^0 = t, x^1, x^2, x^3\) be the transported coordinates on \([t_0, t_*) \times \mathcal{O}\), obtained by following the integral curves of \(T\). Under these coordinates the metric \(g\) takes the form

\[ g = -n^2 dt^2 + g_{ij} dx^i dx^j. \]
Moreover, relative to these coordinates $t, x^1, x^2, x^3$ there hold the evolution equations
\begin{align}
\partial_t g_{ij} &= -2n k_{ij}, \\
\partial_t k_{ij} &= -\nabla_i \nabla_j n + n(R_{ij} + \text{Tr} k_{ij} - 2k_{in}k^0_{nj})
\end{align}
and the constraint equations
\begin{align}
R - |k|^2 + (\text{Tr} k)^2 &= 0, \\
\nabla^j k_{ji} - \nabla_i \text{Tr} k &= 0
\end{align}
on each $\Sigma_t$, where $R_{ij}$ and $R$ denote the Ricci curvature and the scalar curvature of the induced metric $g$ on $\Sigma_t$, and $\text{Tr} k$ denotes the trace of $k$, i.e. $\text{Tr} k = g^{ij} k_{ij}$.
Since $\text{Tr} k = t$ on $\Sigma_t$, it follows from the above equations that
\begin{align}
\text{div} k &= 0 \\
-\Delta n + |k|^2 n &= 1
\end{align}
on each $\Sigma_t$.

The first important breakdown criterion was given by M. Anderson in [2], who showed that when a breakdown occurs at $t^* > 0$ there holds
\[ \limsup_{t \to t^-} \|R\|_{L^\infty(\Sigma_t)} = \infty, \]
where $R$ denotes the Riemannian curvature tensor of the space-time $(M, g)$. Here the pointwise norm $|R|$ is defined with respect to the Riemannian metric $g_T$ on $M$, where $g_T$ is defined as follows: for any $X, Y \in T \Sigma_t$, write $X = X^0 T + \underline{X}$ and $Y = Y^0 T + \underline{Y}$ with $X, Y \in T \Sigma_t$, then
\[ g_T(X, Y) = X^0 Y^0 + g(\underline{X}, \underline{Y}). \]
The result of Anderson implies that if
\[ \sup_{t \in [t_0, t^*)} \|R\|_{L^\infty(\Sigma_t)} = \Lambda_0 < \infty \]
for all $t^* < 0$, then the CMC foliation exists for all values in $[t_0, 0)$.

Recently, Klainerman and Rodnianski [12] provided a new breakdown criterion which shows that if a breakdown happens at $t^* < 0$ then
\[ \limsup_{t \to t^-} (\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}) = \infty, \]
or, in other words, the CMC foliation can be extended beyond any value $t^* < 0$ for which
\[ \sup_{t \in [t_0, t^*)} (\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}) = \Lambda_0 < \infty. \]
In contrast to the breakdown criterion of Anderson, the condition [1.9] of Klainerman and Rodnianski is formally weaker as it refers only to the second fundamental form $k$ and the lapse function $n$ which requires one degree less of differentiability. Moreover, by purely elliptic estimates, one can see that [1.8] implies immediately [1.9], since the boundedness of $\|R\|_{L^\infty}$ exhausts all the dynamical degrees of freedom of the equations. Therefore, the result in [12] is a significant improvement.
We remark that the result of Klainerman and Rodnianski can not be established by purely elliptic estimates. Instead, the proof relies heavily on the tools from the theory of hyperbolic equations. The analogous result has been extended to non-vacuum space-time in [15].

If we consider the Einstein equation expressed relative to the wave coordinates, by energy estimates one can see that the breakdown does not occur unless
\[
\int_{t_0}^{t_*} \|\partial_\tau g\|_{L^\infty} \, dt = \infty.
\]
This condition however is not geometric since it depends on the choice of a full coordinate system. Observe that the components of the second fundamental form \(k\) and \(\nabla \log n\) can be viewed as part of the components of \(\partial_\tau g\). It is natural to ask if we have an integral form of breakdown criterion involving \(k\) and \(n\) only. The first main result of the present paper confirms this and provides a geometric counterpart of (1.10), which can be viewed as an improved version of the breakdown criterion of Klainerman and Rodnianski.

**Theorem 1.1** (Main theorem I). Let \((\mathcal{M}_*, \mathfrak{g})\) be a globally hyperbolic development of \(\Sigma_0\) foliated by the CMC level hypersurfaces of a time function \(t < 0\). Then the space-time together with the foliation \(\Sigma_t\) can be extended beyond any value \(t_* < 0\) for which,
\[
\int_{t_0}^{t_*} \left( \|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)} \right) \, dt = K_0 < \infty.
\]

Let us fix the convention for the deformation tensor of \(\mathbb{T}\), expressed relative to an orthonormal frame \(\{e_0 = \mathbb{T}, e_1, e_2, e_3\}\), as follows,
\[
\pi_{\alpha\beta} = -g(D e_\alpha \mathbb{T}, e_\beta), \quad \text{with } \alpha, \beta = 0, 1, 2, 3.
\]
It is easy to check
\[
\pi_{00} = 0, \quad \pi_{0i} = -\nabla_i \log n, \quad \pi_{ij} = k_{ij}, \quad \text{with } i, j = 1, 2, 3.
\]
Consequently, the condition (1.9) can be formulated as
\[
\sup_{t \in [t_0, t_*]} \|\pi\|_{L^\infty(\Sigma_t)} = \Lambda_0 < \infty,
\]
while the weaker condition (1.11) can be formulated as
\[
(A1) \quad \|\pi\|_{L^1_t L^\infty_{\mathbb{T}}(\mathcal{M}_*)} := \int_{t_0}^{t_*} \|\pi\|_{L^\infty(\Sigma_t)} \, dt = \mathcal{K}_0 < \infty.
\]

We basically follow the framework in [12] to prove Theorem 1.1, however, a sequence of difficulties occur due to the weaker condition (1.11). In order to continue the foliation, according to the local existence theorem given in [5 Theorem 10.2.1], one must establish a global uniform bound for the curvature tensor \(\mathbb{R}\) and \(L^2\)-bounds for its first two covariant derivatives. Since \((\mathcal{M}, \mathfrak{g})\) is a vacuum space-time, by virtue of the Bianchi identity \(\mathbb{R}\) verifies a wave equation of the form
\[
\Box_\mathfrak{g} \mathbb{R} = R \ast R,
\]
where \(\Box\) denotes the covariant wave operator \(\Box = \mathbb{D}^0 \mathbb{D}_\mathfrak{g}\). Based on higher energy estimates it is standard to show that the \(L^2\) bounds for \(\mathbb{D} \mathbb{R}\) and \(\mathbb{D}^2 \mathbb{R}\) can be

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1 Our method applies equally well to the case that \(\Sigma_t\) are asymptotically flat and maximal, i.e \(\text{Tr} k = 0\) and can also be extended to Einstein space-time with matter.
bounded in terms of the $L^\infty$ norm of $R$. Thus, the derivation of the $L^\infty$ bound of $R$ is a crucial step. In order to achieve this goal, Klainerman and Rodnianski [10] succeeded in representing $R(p)$, for each $p \in M_*$, by a Kirchoff-Sobolev formula of the form

$$R(p) = -\int_{\mathcal{N}^-(p, \tau)} A \cdot (R \ast R) + \text{other terms}$$

where $A$ is a 4-covariant tensor defined as a solution of a transport equation along $\mathcal{N}^-(p, \tau)$ with appropriate initial data at the vertex $p$, $\mathcal{N}^-(p, \tau)$ denotes the portion of the null boundary $\mathcal{N}^-(p)$ in the time interval $[t(p) - \tau, t(p)]$. The past null cone $\mathcal{N}^-(p)$ is in general an achronal Lipschitz hypersurface ruled by the set of past null geodesics from $p$. In order to derive all necessary estimates, one must show that $\mathcal{N}^-(p)$ remains a smooth hypersurface in the time slab $[t(p) - \tau, t(p)]$ for some universal constant $\tau > 0$. Therefore, it is necessary to provide a uniform lower bound for the past null radius of injectivity at all $p \in M_*$.

Let us recall briefly the definition of the past null radius of injectivity at $p$, one may consult [11] for more details. We parametrize the set of past null vectors in $T_pM$ in terms of $\omega \in S^2$, the standard sphere in $\mathbb{R}^3$. Then, for each $\omega \in S^2$, let $l_\omega$ be the null vector in $T_pM$ normalized with respect to the future, unit, time-like vector $T_p$ by

$$g(l_\omega, T_p) = 1$$

and let $\Gamma_\omega(s)$ be the past null geodesic with initial data $\Gamma_\omega(0) = p$ and $\frac{dl_\omega}{ds}(0) = l_\omega$.

We define the null vector field $L$ on $\mathcal{N}^-(p)$ by

$$L(\Gamma_\omega(s)) = \frac{d}{ds} \Gamma_\omega(s)$$

which may only be smooth almost everywhere on $\mathcal{N}^-(p)$ and can be multi-valued on a set of exceptional points. We can choose the parameter $s$ with $s(p) = 0$ so that

$$D_L L = 0 \quad \text{and} \quad L(s) = 1.$$ 

This $s$ is called the affine parameter.

The past null radius of injectivity $i_*(p)$ at $p$ is then defined to be the supremum over all the values $s_0 > 0$ for which the exponential map

$$\mathcal{g}_p : (s, \omega) \rightarrow \Gamma_\omega(s)$$

is a global diffeomorphism from $(0, s_0) \times S^2$ to its image in $\mathcal{N}^-(p)$. It is known that $i_*(p) > 0$ for each $p$, $\mathcal{N}^-(p)$ is smooth within the null radius of injectivity, and

$$i_*(p) = \min\{s_*(p), l_*(p)\},$$

where $s_*(p)$, the past null radius of conjugacy at $p$, is defined to be the supremum over all values $s_0 > 0$ such that the exponential map $\mathcal{g}_p$ is a local diffeomorphism from $(0, s_0) \times S^2$ to its image in $\mathcal{N}^-(p)$, and $l_*(p)$, the past cut locus radius at $p$, is defined to be the smallest value of $s_0$ for which there exist two distinct null geodesics $\Gamma_1$ and $\Gamma_2$ from $p$ with $\Gamma_1(s_0) = \Gamma_2(s_0)$. Thus, for a past null geodesic $\Gamma_\omega$ from $p$, a point $q = \Gamma_\omega(s_*)$ is called a conjugate point of $p$ if $\mathcal{g}_p$ is singular at $(s_*, \omega)$, while it is called a null cut point of $p$ if $\mathcal{g}_p$ is nonsingular at $(s_*, \omega)$ and through $q$ there exists another null geodesic emanating from $p$.

Since we are working on the CMC foliation, it is convenient to introduce the past null radius of injectivity $i_*(p, t)$ at each $p$ with respect to the global time function
We define \( i_*(p, t) \) to be the supremum over all the values \( \tau > 0 \) for which the exponential map
\[
(1.13) \quad G_p : (t, \omega) \rightarrow \Gamma_\omega(s(t))
\]
is a global diffeomorphism from \( (t(p) - \tau, t(p)) \times S^2 \) to its image in \( \mathcal{N}^-(p) \). We remark that \( s \) is a function not only depending on \( t \) but also on \( \omega \), we suppress \( \omega \) just for convenience. It is known that
\[
i_*(p, t) = \min \{ s_*(p, t), l_*(p, t) \},
\]
where \( s_*(p, t) \) is defined to be the supremum over all values \( \tau > 0 \) such that the map \( G_p \) is a local diffeomorphism from \( (t(p) - \tau, t(p)) \times S^2 \) to its image, and \( l_*(p, t) \) is defined to be the smallest value of \( \tau > 0 \) for which there exist two distinct null geodesics \( \Gamma_1(s(t)) \) and \( \Gamma_2(s(t)) \) from \( p \) which intersect at a point with \( t = t(p) - \tau \).

In [11] Klainerman and Rodnianski provided a uniform lower bound on the null radius of injectivity under the assumption (1.19). In order to complete the proof of Theorem 1.1, one must provide a uniform lower bound on the null radius of injectivity under the weaker condition (1.11). This is contained in the second main result of the present paper.

**Theorem 1.2** (Main theorem II). Assume that \( \mathcal{M}_* \) is a globally hyperbolic development of \( \Sigma_0 \) verifying the condition (1.17). Then for all \( p \in \mathcal{M}_* \) there holds
\[
(1.14) \quad i_*(p, t) > \min \{ \delta_*, t(p) - t_0 \},
\]
where \( \delta_* > 0 \) is a constant depending only on \( Q_0, \mathcal{K}_0, |\Sigma_0| \) and \( t_* \).

In order to prove this result, it is useful to review the essential steps in the work of Klainerman and Rodnianski in [11]. The first step is to show that
\[
(1.15) \quad s_*(p, t) > \min \{ l_*(p, t), \delta_* \}
\]
for some universal constant \( \delta_* > 0 \). This can be achieved by showing that
\[
(1.16) \quad \sup_{\mathcal{N}^-(p, \tau)} \left| \text{tr} \chi - \frac{2}{s(t)} \right| \leq C
\]
with \( \tau := \min \{ l_*(p, t), \delta_* \} \), where \( \chi \) is the null second fundamental form \( \chi_{AB} = g(D_A L, e_B) \) of the 2-dimensional space-like surface \( S_t := \mathcal{N}^-(p) \cap \Sigma_t \) with \( (e_A)_{A=1,2} \) being a frame field tangent to \( S_t \). The analog has been carried out in [7, 8, 9, 13] for geodesic foliations under the boundedness assumption of the curvature flux. In order to adapt those arguments to prove (1.16) for the time foliations, one needs to show that \( t(p) - t \) and \( s \) are comparable and the geodesic curvature flux (see [11]) is bounded, both of which rely on the relation
\[
(1.17) \quad |a - 1| \leq \frac{1}{2} \quad \text{on} \quad \mathcal{N}^-(p, \tau),
\]
where \( a \), the null lapse function, is defined by \( a^{-1} := g(T, L) \) with \( a(p) = 1 \). Note that along a null geodesic
\[
\frac{dt}{ds} = -(an)^{-1}, \quad \frac{da}{ds} = \nu, \quad \nu := k_{NN} - \nabla_N \log n,
\]
\( Q_0 \) denotes the Bel-Robinson energy on the initial slice \( \Sigma_0 \) which will be defined in Section 2.

\( A \) universal constant always means a constant depending only on \( Q_0, \mathcal{K}_0, |\Sigma_0|, t_* \) and the number \( t_0 > 0 \) such that \( t_0^{-1} \leq (g_{ij}) \leq t_0 \) on the initial slice \( \Sigma_0 \). Throughout this paper \( C \) always denotes a universal constant.
where $N$ is the unit inward normal of $S_t$ in $\Sigma_t$. If (1.9) is satisfied, one can see that (1.17) holds for $t(p) - \delta^* \leq s \leq t(p)$ for some universal $\delta^* > 0$, and consequently $s$ and $t(p) - t$ are comparable. However, under the weaker condition (1.11) only, it is highly nontrivial to obtain (1.17). We observe that (1.17) can be achieved by establishing

$$\|\nu\|^2_{L^2(N^-(p,\tau))} = \sup_{\omega \in S^2} \int_{\Gamma_\omega} an|\nu|^2dt \leq C$$

where $\Gamma_\omega$ is the portion of a past null geodesic initiating from $p$ contained in $N^-(p,\tau)$, for some universal constant $\delta^* > 0$. How to obtain such an estimate on $\nu$ is the first difficulty we encounter. The idea to derive the trace estimate (1.18) is to employ the techniques in the proof of the sharp trace inequality in [7, 8, 9]. Under the assumption (1.11) only, suppose the sharp trace inequality holds true on null cone in time foliation, in order to prove (1.18), schematically, we need to prove

i) there holds for $/\nabla \nu$ the decomposition

$$/\nabla \nu = \nabla_L P + Q$$

with $P$ and $Q$ appropriate $S_t$ tangent tensors.\footnote{$/\nabla$ denotes the connection with respect to the induced metric $\gamma$ on $S_t$.}

ii) there holds

$$\|/\nabla (\nu, P)\|_{L^2(N^-(p,\tau))} + \|\nabla_L (\nu, P)\|_{L^2(N^-(p,\tau))} \leq C.$$

The decomposition of the form (1.19) will be derived in [14]. To prove the sharp trace inequality in time foliation and to control $P$ and $Q$ must be coupled with the proof of a series of estimates for the Ricci coefficients on null hypersurface $N^-(p,\tau)$ including (1.16) by a delicate bootstrap argument. Hence, under the condition (1.11) only, (1.16), (1.17) and (1.18) should be proved simultaneously. The proof is rather involved and close to the spirit of the works [7, 8, 9, 13]. We will present it in [14] with full details.

Now we simply consider how to obtain the estimate for $\nu$ in (1.20). The estimate for $\nabla_N \log n$ of the form (1.20) can be obtained by elliptic estimates and trace inequality. By elliptic estimate, in view of

$$\text{div} k = 0, \text{curl} k = H,$$

where $H$ denotes the magnetic part of $R$, we can only derive $\|k\|_{H^1(\Sigma)} \leq C$, which, by classic trace theorem, loses $1/2$ derivative if restricted to null cone. However, (1.20) requires the $L^2$ control of one derivative of $k_{NN}$ on null cones. Hence, we must adopt a different approach, which significantly surpasses the one via elliptic estimate and trace inequality. This inspires us to use the tensorial wave equation for $k$, which symbolically is given by

$$\Box k = k \cdot Ric + n^{-2}\nabla^2 \dot{n} + \pi \cdot \nabla k - n^{-3} n \nabla^2 n + \pi \cdot \pi \cdot \pi = 0.$$  

We then prove by energy method, the $k$-flux satisfies

$$\|/\nabla k\|_{L^2(N^-(p,\tau))} + \|\nabla_L k\|_{L^2(N^-(p,\tau))} \leq C,$$

which schematically gives the desired control on $k_{NN}$. The next step is to find a system of good local space-time coordinates under which $g$ is comparable with the Minkowski metric. More precisely, for a sufficiently small constant $\epsilon > 0$, one needs to show that there exists a constant $\delta^* > 0$, 

\footnote{\(/\nabla\) denotes the connection with respect to the induced metric $\gamma$ on $S_t$.}
depending only on $\varepsilon$ and some universal constants, for which each geodesic ball $B_\delta(p)$ with $p \in \Sigma_t$ admits local coordinates $x = (x^1, x^2, x^3)$ such that under the corresponding transport coordinates $x^0 = t, x^1, x^2, x^3$ the metric $g$ has the expression (1.1) with

\[(1.24) \quad |n - n(p)| \leq \varepsilon \quad \text{and} \quad |g_{ij} - \delta_{ij}| \leq \varepsilon\]

on $B_\delta(p) \times [t(p) - \delta_*, t(p)]$. The existence of such local coordinates together with \[(1.16)\] will enable us to show that $\mathcal{N}^-(p, \delta_*)$ is close to the flat cone and consequently $l^*(p, t) \geq \delta_*$. The part on $n$ in (1.24) can be established by elliptic estimates on $n$ and $\partial_t n$. The derivation of the result for $g$ under the weaker condition (1.11), however, presents one of the core difficulties, which invokes new methods and a second application of (1.22).

By the Bel-Robinson energy bound $Q(t) \leq C$ and a result of Anderson [1], one can control the lower bound of harmonic radius on $\Sigma_t$, such that with the coordinates $x = (x^1, x^2, x^3)$ on $B_\delta(p) \subset \Sigma_t$,

\[|g_{ij}(x, t(p)) - \delta_{ij}| \leq \frac{1}{2} \varepsilon.\]

The challenge is to control time evolution of $g$. Using (1.2), one has

\[(1.25) \quad |g_{ij}(x, t(p)) - g_{ij}(x, t)| \lesssim \int_t^{t(p)} |k(x, t')| dt'.\]

If (1.9) holds, or more generally, if

\[\int_{t_0}^{t} \|k(t')\|_{L^q(\Sigma_t')} dt' \leq \Lambda_0 < \infty\]

for some $q > 1$, then with $\delta_*$ sufficiently small

\[(1.26) \quad |g_{ij}(x, t(p)) - g_{ij}(x, t)| \leq \Lambda_0^{1/q} (t(p) - t)^{1-1/q} < \frac{1}{2} \varepsilon.\]

The above argument fails if $k$ verifies (1.11) only. Under the assumption (1.11), our strategy is to prove directly the integral on the right of (1.24) can be small, i.e.

\[\int_t^{t(p)} |k(x, t')| dt' < \frac{1}{2} \varepsilon, \quad \forall x \in \Sigma\]

by establishing

\[(1.27) \quad \sup_{x \in \Sigma} \int_t^{t(p)} |k(x, t')|^2 dt' \leq C,\]

since

\[|g_{ij}(x, t(p)) - g_{ij}(x, t)| \lesssim \left( \int_t^{t(p)} |k(x, t')|^2 dt' \right)^{1/2} (t(p) - t)^{1/2} \lesssim (t(p) - t)^{1/2}\]

which implies $|g_{ij}(x, t(p)) - g_{ij}(x, t)| < \frac{1}{2} \varepsilon$ as long as $\delta_*$ is appropriately chosen.

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5We use $\Phi_1 \leq \Phi_2$ to mean that $\Phi_1 \leq C \Phi_2$ for some universal constant $C$. 
The major part of the present paper is therefore to establish (1.27) under the weaker condition (1.11). To this end, we will use the Kirchoff parametrix to represent \( k(p) \) as
\[
-4\pi n(p)k(p) \cdot J = \int_{\mathbb{N}^{-}(p,\tau)} \Box k \cdot A + \text{other terms},
\]
for any \( \delta < i_{*}(p,t) \), where \( J \) is any 2-covariant tensor at \( p \) tangent to \( \Sigma_{t(p)} \) and \( A \) is the \( \Sigma \)-tangent tensor defined by
\[
D_{L}A_{ij} + \frac{1}{2}\text{tr} \chi A_{ij} = 0 \quad \text{on} \quad \mathbb{N}^{-}(p,\tau), \quad \lim_{t \to t(p)} (t(p) - t) A_{ij} = J.
\]
It can be shown that \( \|rA\|_{L^{\infty}(\mathbb{N}^{-}(p,\tau))} \lesssim 1 \) together with other estimates on \( A \), where \( r = \sqrt{(4\pi)^{-1}|S_{t}|} \) and \( |S_{t}| \) denotes the area of \( S_{t} \). Thus
\[
n(p)|k(p)| \lesssim \int_{\mathbb{N}^{-}(p,\tau)} r^{-1}\Box k + \text{other terms}.
\]
Next we let \( p \) move along an integral curve \( \Phi(t) \) of \( T \) to get the representations of \( k \) at all points on this curve. Then we can reduce the proof of (1.27) to showing that
\[
\int_{t(p) - \tau}^{t(p)} \left( \int_{\mathbb{N}^{-}(\Gamma(t),t-t(p)+\tau)} r^{-1}\Box k + \cdots \right)^{2} dt \lesssim 1.
\]
In view of (1.22), we have to employ various estimates of \( k \) and \( n \) on the null cones, which will be established by delicate analysis.

This paper is organized as follows. In Section 2 we collect some preliminary results related to the CMC foliation, which will be used frequently in the later sections. In Section 3 we establish various elliptic estimates on the lapse function \( n \), in particular, we show that \( n \) can be bounded from below and above by positive universal constants. In Section 4 we provide the sketch of the proof of Theorem 1.2. We will explain how to use the bootstrap argument to establish (1.16) and other related estimates on the null cones. We then show how to use the estimate (1.27) to obtain a system of good local space-time coordinates which is crucial for completing the proof of Theorem 1.2. In order to establish (1.27), we derive a tensorial wave equation for \( k \) in Section 5 and provide the estimate for the so called \( k \)-flux in Section 6 which will be defined later. In Section 7 we provide some trace estimates on the surfaces \( S_{t} \). We then use these results in Section 8 to establish various estimates for \( k, n \) and \( \chi \) on the null cones. In section 9 we adapt the Kirchoff-Sobolev formula in [10] to represent the second fundamental form \( k \) along the null cones, through which we give the proof of (1.10) under the condition (1.11) and thus complete the proof of Theorem 1.2. Finally in Section 9 we complete the proof of Theorem 1.1.

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2. Preliminaries

For the lapse function $n$, by using the elliptic equation $-\Delta n + |k|^2 n = 1$, it follows easily from the maximum principle that

$$(2.1) \quad \frac{1}{\|k\|_{L^\infty(\Sigma_t)}} \leq n \leq \frac{3}{t^2} \quad \text{on } \Sigma_t.$$ 

Thus, if we knew that $\|k\|_{L^\infty(\Sigma_t)}$ is uniformly bounded with respect to $t \in [t_0, t_\ast)$, then we could get a positive uniform lower bound on $n$. Unfortunately, we only have the weaker assumption (A1) on $k$, which does not allow (2.1) to give a positive uniform lower bound on $n$ directly. In the next section, we will show under the assumption (A1) that $C^{-1} \leq n \leq C$ on $\mathcal{M}_f$ for some universal constant $C > 0$.

For each slice $\Sigma_t$, we use $|\Sigma_t|$ to denote its volume. Then, by using $\partial_t g_{ij} = - 2nk_{ij}$ and $\text{Tr} k = t$ on $\Sigma_t$ we have

$$\frac{d}{dt} \left( |t|^3 |\Sigma_t| \right) = \int_{\Sigma_t} t^2 \left( t^2 n - 3 \right) d\mu_g \leq 0.$$ 

This implies that $|t|^3 |\Sigma_t|$ is decreasing with respect to $t$. Consequently

$$(2.2) \quad |\Sigma_t| \leq \frac{|t_0|^3}{|t|^3} |\Sigma_{t_0}|, \quad \forall t \leq t_\ast.$$ 

2.1. Bel-Robinson Energy. We start with a brief review of Bel-Robinson energy, one may consult [5] for more details. Associated to the Weyl tensor $R$, the Bel-Robinson tensor is the full symmetric, traceless tensor defined by

$$(2.3) \quad Q[R]_{\alpha\gamma\beta\delta} = R_{\alpha\lambda\gamma\mu}R^{\lambda\delta\mu\beta} + *R_{\alpha\lambda\gamma\mu}^\ast R^{\lambda\delta\mu\beta}.$$ 

Then $Q[R](X, Y, X, Y) \geq 0$ whenever $X, Y$ are timelike vectors, with equality only if $R = 0$. Let $P_\alpha = Q[R]_{\alpha\beta\gamma\delta} T^\beta T^\gamma T^\delta$. Since $R_{\alpha\beta} = 0$, a straightforward calculation shows that

$$(2.4) \quad D^\alpha P_\alpha = -3\pi^{\alpha\beta} Q[R]_{\alpha\beta\gamma\delta} T^\gamma T^\delta.$$ 

If we introduce the Bel-Robinson energy $Q(t)$ by

$$Q(t) := \int_{\Sigma_t} Q[R](T, T, T, T) d\mu_{\Sigma_t},$$ 

then, by integrating (2.4) in a slab $\mathcal{M}_f = \cup_{t \in J} \Sigma_t$ with $J = [t_0, t] \subset [t_0, t_\ast)$, we obtain

$$Q(t) = Q(t_0) - 3\int_{t_0}^t \int_{\Sigma_t} nQ[R]_{\alpha\beta\gamma\delta} \pi^{\alpha\beta} d\mu_{\Sigma_t} dt'.$$ 

Let $E$ and $H$ denote the electric and magnetic parts of the curvature tensor $R$ defined by

$$(2.5) \quad E(X, Y) = g(R(X, T)T, Y), \quad H(X, Y) = g(*R(X, T)T, Y)$$

with $*R$ the Hodge dual of $R$. It is well known that $E$ and $H$ are traceless symmetric 2-tensors tangent to $\Sigma_t$ with

$$|R|^2 = |E|^2 + |H|^2,$$

$$|Q| \leq 4(|E|^2 + |H|^2)$$

and

$$Q(T, T, T, T) = |E|^2 + |H|^2.$$
Therefore
\[ Q(t) \leq Q(t_0) + 12 \int_{t_0}^t \|n\pi\|_{L^\infty(\Sigma_t')} Q(t') dt'. \]

By the Gronwall inequality it follows that
\[ Q(t) \leq Q(t_0) \exp\left(12 \int_{t_0}^t \|n\pi\|_{L^\infty(\Sigma_t')} dt'\right) \]
for all \( t \in [t_0, t_\ast) \). Therefore, in view of the condition \((A1)\) we obtain the uniform boundedness of the Bel-Robinson energy.

**Lemma 2.1.** Under the condition \((A1)\), there exists a constant \( C \) depending only on \( K_0 \) and \( t_\ast \) such that
\[ Q(t) \leq CQ_0^2 \]
for all \( t \in [t_0, t_\ast) \), where \( Q_0^2 := Q(t_0) \).

Consequently we have

**Lemma 2.2.** Let the condition \((A1)\) hold. Then on any CMC leaf \( \Sigma_t \subset M_\ast \) there holds
\[ \int_{\Sigma_t} \left( |\nabla k|^2 + \frac{1}{4} |k|^4 \right) + \int_{\Sigma_t} |\text{Ric}|^2 \lesssim Q_0^2. \]

**Proof.** The inequality on \( k \) follows from [12, Proposition 8.4] and Lemma 2.1. The inequality on \( \text{Ric} \) then follows from the identity \( R_{ij} - k_{ia}k^a_j + \text{Tr}k_{ij} = E_{ij} \).

2.2. Harmonic coordinates. For any coordinate chart \( O \subset \Sigma_0 \) with local coordinates \( x = (x^1, x^2, x^3) \), we denote by \( x^0 = t, x^1, x^2, x^3 \) the transported coordinates on \( I \times O \) obtained by transporting along the integral curves of \( T \). The following is an immediate consequence of \((A1)\) and \([12]\).

**Proposition 1.** Let the assumption \((A1)\) hold. There exists a positive constant \( C_0 \) depending only on \( K_0 \) such that, relative to the induced transported coordinates \( x^0 = t, x^1, x^2, x^3 \) in \( I \times O \) we have
\[ C_0^{-1} |\xi|^2 \leq g_{ij}(t, x)\xi^i\xi^j \leq C_0 |\xi|^2. \]

**Proof.** This is [12, Proposition 2.4] which was stated under the stronger condition \([13]\), the proof however requires only the weaker assumption \((A1)\).

This proposition enables us to derive a uniform lower bound on the volume radius for all the slices \( \Sigma_t \). Here, for a 3-dimensional Riemannian manifold \((M, g)\), the volume radius \( r_{vol}(p, \rho) \) at a point \( p \in M \) and scales \( \rho \) is defined by
\[ r_{vol}(p, \rho) = \inf_{r \leq \rho} \frac{|B_r(p)|}{r^3} \]
with \(|B_r(p)|\) the volume of \( B_r(p) \) relative to metric \( g \). The volume radius \( r_{vol}(M, \rho) \) of \( M \) on scales \( \rho \) is the infimum of \( r_{vol}(p, \rho) \) over all \( p \in M \). Using Proposition 1 it has been show in [11] Proposition 4.4 that the volume radius \( r_{vol}(\Sigma_t, 1) \) of each \( \Sigma_t \) on scales \( \leq 1 \) verifies
\[ r_{vol}(\Sigma_t, 1) \geq v_0 \]
for some constant \( v_0 > 0 \) depending only on \( K_0 \).
From the previous subsection we have already obtained, under (A1), that
\[
\|Ric\|_{L^2(\Sigma_t)} \leq C \quad \text{and} \quad |\Sigma_t| \leq \left[ \frac{t_0}{|t_*|^2} \right]^{3/2} |\Sigma_{t_0}|.
\]
Therefore, Theorem 3.5 in \cite{1} applies and provides the following results on the existence of harmonic coordinates.

**Proposition 2.** Let the assumption (A1) hold. For any \( \epsilon > 0 \), there exists \( r_0 > 0 \) depending on \( \epsilon, Q_0, K_0, |\Sigma_0| \) and \( t_* \) such that every geodesic ball \( B_r(p) \subset \Sigma_t \) with \( r \leq r_0 \) admits a system of harmonic coordinates \( x = (x^1, x^2, x^3) \) under which
\[
(1 + \epsilon)^{-1} \delta_{ij} \leq g_{ij} \leq (1 + \epsilon) \delta_{ij}
\]
\[
r \int_{B_{r}(p)} |\partial^2 g_{ij}|^2 d\mu \leq \epsilon.
\]

We will not use the full strength of this result. The crucial part in our applications is the existence of a local coordinates \( x = (x^1, x^2, x^3) \) on each \( B_{r_0}(p) \subset \Sigma_t \) satisfying (2.8) with \( r_0 > 0 \) depending only on \( \epsilon, Q_0, K_0, |\Sigma_0| \) and \( t_* \).

### 2.3. Sobolev-type inequalities

We will give several Sobolev type inequalities under the assumption (A1). These inequalities are useful in establishing various estimates.

**Lemma 2.3.** Let the assumption (A1) hold on \( M_* \). Then for any smooth tensor field \( F \) on \( \Sigma_t \subset M_* \) and any \( 2 \leq p \leq 6 \) there holds
\[
\|F\|_{L^p(\Sigma_t)} \leq C \left( \|\nabla F\|_{L^2(\Sigma_t)}^{3/2 - 3/p} \|F\|_{L^2(\Sigma_t)}^{3/p - 1/2} + \|F\|_{L^2(\Sigma_t)} \right),
\]
where \( C \) is a constant depending only on \( K_0 \) and \( p \).

**Proof.** This is \cite{12} Corollary 2.7. \( \square \)

The following calculus inequality is useful in deriving \( L^\infty \) bounds of certain quantities.

**Lemma 2.4.** Let the assumption (A1) hold on \( M_* \). Then for any smooth tensor field \( F \) on \( \Sigma_t \subset M_* \) and \( 3 < p \leq 6 \) there holds
\[
\|F\|_{L^\infty(\Sigma_t)} \leq C \left( \|\nabla^2 F\|_{L^2(\Sigma_t)}^{3/2 - 3/p} \|\nabla F\|_{L^2(\Sigma_t)}^{3/p - 1/2} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)} \right),
\]
where \( C \) is a constant depending only on \( K_0 \) and \( p \).

**Proof.** By using a partition of unity, the Sobolev embedding \( W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \) with \( p > 3 \), and (2.7) in Proposition \cite{1} it is easy to derive for any scalar function \( f \) on \( \Sigma_t \) that
\[
\|f\|_{L^\infty(\Sigma_t)} \leq C \left( \|\nabla f\|_{L^p(\Sigma_t)} + \|f\|_{L^p(\Sigma_t)} \right).
\]
Now we take \( f = |F|^2 \) in the above inequality. It yields
\[
\|F\|_{L^\infty(\Sigma_t)}^2 \leq C \left( \|\nabla |F|^2\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)}^2 \right) \leq C \left( \|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)} \right) \|F\|_{L^\infty(\Sigma_t)}.
\]
This implies for \( p > 3 \) that
\[
\|F\|_{L^\infty(\Sigma_t)} \leq C \left( \|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)} \right).
\]
The desired inequality follows by applying Lemma 2.3 to the term \( \|\nabla F\|_{L^p(\Sigma_t)}. \) \( \square \)
3. Elliptic estimates for the lapse function $n$

In this section, we establish a series of elliptic estimates on the lapse function $n$ together with $n^{-1}$ and $\dot{n} := \partial_t n$ under the assumption (A1). These results will be repeatedly used in later sections. Throughout this paper we will use $C$ to denote a universal constant.

3.1. Estimates on $n$.

**Proposition 3.** Let the assumption (A1) hold. Then on every $\Sigma_t \subset \mathcal{M}_*$ there holds

$$\|\nabla^2 n\|_{L^2(\Sigma_t)} + \|\nabla n\|_{L^2(\Sigma_t)} \leq C.$$

**Proof.** We multiply the equation $-\Delta n + \|k|^2 n = 1$ by $n$ and integrate over $\Sigma_t$ to obtain

$$\int_{\Sigma_t} (|\nabla n|^2 + |k|^2 n^2) = \int_{\Sigma_t} n.$$

Since $0 < n \leq 3/t^2 \leq 3/t^2$ and $|\Sigma_t| \leq |\Sigma_0| |t_0|^3/|t|^4$, this immediately gives the desired bound on $\|\nabla n\|_{L^2(\Sigma_t)}$.

In order to obtain the bound on $\|\nabla^2 n\|_{L^2(\Sigma_t)}$, we use the Böchner identity

$$\int_{\Sigma_t} |\nabla^2 n|^2 = \int_{\Sigma_t} (|\Delta n|^2 - R_{ij} \nabla^i n \nabla^j n),$$

the equation $\Delta n = |k|^2 n - 1$, Lemma 2.2 and the Hölder inequality to infer that

$$\|\nabla^2 n\|_{L^2} \lesssim \|k\|_{L^2}^2 + |\Sigma_t|^2 + \|\text{Ric}\|_{L^2}^2 \|\nabla n\|_{L^4} \lesssim 1 + \|\nabla n\|_{L^4}.$$

With the help of Lemma 2.2 we have

$$\|\nabla^2 n\|_{L^2} \lesssim 1 + \|\nabla^2 n\|_{L^2}^{3/4} \|\nabla n\|_{L^4}^{1/4} + \|\nabla n\|_{L^2}.$$

Therefore

$$\|\nabla^2 n\|_{L^2} \lesssim 1 + \|\nabla n\|_{L^2} \lesssim 1$$

and the proof is complete. \qed

**Proposition 4.** Let the assumption (A1) hold. Then there hold

\begin{align*}
(3.1) &\quad \|\nabla^3 n\|_{L^1 L^2(\mathcal{M}_*)} \leq C \\
(3.2) &\quad \|\nabla n\|_{L^1 L^2(\mathcal{M}_*)} \leq C
\end{align*}

where $1 \leq b < 2$.

We will give the proof with the help of the following lemma.

**Lemma 3.1.** Let the assumption (A1) hold. Then for any 1-form $F$ on $\Sigma_t \subset \mathcal{M}_*$ we have

\begin{equation}
(3.3) \quad \|\nabla^2 F\|_{L^2(\Sigma_t)} \leq C \left( \|\Delta F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)} \right).
\end{equation}

**Proof.** It is well known that for any 1-form $F$ on $\Sigma_t$ there holds the Böchner identity

\begin{align*}
(3.4) \quad &\int_{\Sigma_t} |\Delta F|^2 = \int_{\Sigma_t} |\nabla^2 F|^2 - \frac{1}{2} \int_{\Sigma_t} R_{d\text{iac}} R_{m\text{iac}} F_d F_m \\
&\quad + \int_{\Sigma_t} R_{ad} \nabla_d F_i \nabla_a F_i - \int_{\Sigma_t} R_{id\text{ac}} \nabla_c F_d \nabla_a F_i.
\end{align*}
2.4 to obtain the estimate
Thus, we may use (3.4), the Hölder inequality, Lemma 2.2, Lemma 2.3 and Lemma 2.4 to obtain the estimate
\[ \|\nabla^2 F\|_{L^2} \lesssim \|\Delta F\|_{L^2} + \|\text{Ric}\|_{L^2}^{1/2} \|\nabla F\|_{L^4} + \|F\|_{L^\infty} \|\text{Ric}\|_{L^2} \]
\[ \lesssim \|\Delta F\|_{L^2} + \left(\|\nabla^2 F\|_{L^2}^{3/4} \|\nabla F\|_{L^2}^{1/4} + \|\nabla F\|_{L^2}\right). \]
With the help of Young’s inequality, the inequality (3.3) follows immediately. \( \square \)

**Proposition 5.** Let the assumption (A1) hold. Then on every \( \Sigma_t \subset M_* \) there hold
\[ (3.5) \quad \|\nabla^3 n\|_{L^2(\Sigma_t)} \leq C \left(\|\nabla n\|_{H^1(\Sigma_t)} + \|k\|_{L^\infty(\Sigma_t)}\right), \]
\[ (3.6) \quad \|\nabla n\|_{L^\infty(\Sigma_t)} \leq C \left(\|\nabla n\|_{H^2(\Sigma_t)} + \|k\|_{L^\infty(\Sigma_t)}^{3/2 - \frac{3}{p} - \frac{1}{2}} \|\nabla^2 n\|_{L^2(\Sigma_t)}^{3/p - 1/2}\right), \]
where \( 3 < p \leq 6 \).

**Proof.** A simple application of Lemma 3.1 to \( F = \nabla n \) gives
\[ (3.7) \quad \|\nabla^3 n\|_{L^2} \leq \|\Delta \nabla n\|_{L^2} + \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^2}. \]
Recall the commutation formula \( \Delta \nabla n = \nabla_i \Delta n + R_{ij} \nabla_j n \) and the equation \( -\Delta n + |k|^2 n = 1 \), we can estimate
\[ \|\Delta \nabla n\|_{L^2} \leq \|k\|_{L^2}^2 \|\nabla n\|_{L^6} + \|k\|_{L^\infty} \|\nabla \nabla n\|_{L^2} + \|\text{Ric}\|_{L^2} \|\nabla n\|_{L^\infty}. \]
Plugging this into (3.7), using Lemma 2.2 and Lemma 2.3 gives
\[ \|\nabla^3 n\|_{L^2} \lesssim \|\nabla n\|_{L^\infty} + \|\nabla n\|_{H^1} + \|k\|_{L^\infty}. \]
Using Lemma 2.4 for the term \( \|\nabla n\|_{L^\infty} \) with \( p = 4 \), we then obtain
\[ \|\nabla^3 n\|_{L^2} \lesssim \|\nabla^3 n\|_{L^2}^{3/4} \|\nabla^2 n\|_{L^2}^{1/4} + \|\nabla n\|_{H^1} + \|k\|_{L^\infty}. \]
This clearly implies (3.5). The inequality (3.6) is an immediate consequence of (3.5) and Proposition 4. \( \square \)

Proposition 5 follows by integrating (3.5) and (3.6) in time with the help of (A1) and Proposition 3.

### 3.2. Estimates on \( n^{-1} \)

**Proposition 6.** Let the assumption (A1) hold. Then on each \( \Sigma_t \subset M_* \) there hold
\[ \|\nabla^2 (n^{-1})\|_{L^2(\Sigma_t)} + \|n^{-1}\|_{L^\infty(\Sigma_t)} \leq C. \]

**Proof.** We first have from the Bochner identity that
\[ \int_{\Sigma} |\nabla^2 (n^{-1})|^2 = \int_{\Sigma} |\Delta (n^{-1})|^2 - \int_{\Sigma} R_{ij} \nabla_i (n^{-1}) \nabla_j (n^{-1}) \]
\[ \leq \|\Delta (n^{-1})\|^2_{L^2} + \|\text{Ric}\|_{L^2} \|\nabla (n^{-1})\|^2_{L^2}. \]
Since \( -\Delta n + |k|^2 n = 1 \), we have
\[ (3.9) \quad \Delta (n^{-1}) = 2n^{-3} |\nabla n|^2 + n^{-2} - |k|^2 n^{-1}. \]
Consequently, it follows from the Hölder inequality that
\[ \| \Delta (n^{-1}) \|_{L^2} \lesssim \| n^{-1} \nabla n \|_{L^4} \| \nabla (n^{-1}) \|_{L^4} + \| k \|_{L^6}^2 \| n^{-1} \|_{L^6} + \| n^{-1} \|_{L^4}^2. \]

Combining this inequality with (3.8) and using the Sobolev embedding \( H^1(\Sigma) \hookrightarrow L^p(\Sigma) \) with \( 2 \leq p \leq 6 \), which is a consequence of Lemma 2.3, we obtain
\[
\| \nabla^2 (n^{-1}) \|_{L^2} \lesssim \| n^{-1} \nabla n \|_{L^4} \| \nabla (n^{-1}) \|_{L^4} + (\| n^{-1} \|_{H^1} + \| k \|_{L^6}^2) \| n^{-1} \|_{H^1}
\]
(3.10)

We need to estimate \( \| n^{-1} \nabla n \|_{L^4} \). To this end, we multiply the equation \(-\Delta n + |k|^2 n = 1\) by \( n^{-1} \) for some positive integer \( l \) and then integrate by parts over \( \Sigma_t \) to obtain
\[
\int_{\Sigma_t} (l n^{-l-1} |\nabla n|^2 + n^{-1}) = \int_{\Sigma_t} n^{-l+1} |k|^2.
\]
Taking \( l = 7 \) gives
\[
\int_{\Sigma_t} n^{-8} |\nabla n|^2 \lesssim \int_{\Sigma_t} n^{-6} |k|^2 \lesssim \| k \|_{L^4}^2 \| n^{-2} \|_{L^6}^3.
\]
Therefore
\[
\| n^{-1} \nabla n \|_{L^4} \leq \left( \int_{\Sigma_t} n^{-8} |\nabla n|^2 \right)^{1/8} \left( \int_{\Sigma_t} |\nabla n|^6 \right)^{1/8} \lesssim \| k \|_{L^4}^{1/4} \| n^{-2} \|_{L^6}^{3/8} \| \nabla n \|_{L^6}^{3/8}.
\]

By Lemma 2.3 and Proposition 3.11, we have \( \| \nabla n \|_{L^6} \leq C (\| \nabla^2 n \|_{L^2} + \| \nabla n \|_{L^2}) \leq C. \)

By using Lemma 2.3 and (3.11) with \( l = 5 \) we also have
\[
\| n^{-2} \|_{L^6} \lesssim \| n^{-2} \|_{H^1} \lesssim \left( \int_{\Sigma_t} n^{-4} |k|^2 \right)^{1/2} + \| n^{-1} \|_{L^4}^2
\]
\[
\lesssim \| k \|_{L^6}^2 \| n^{-1} \|_{L^6}^2 + \| n^{-1} \|_{L^4}^2
\]
\[
\lesssim (1 + \| k \|_{L^6}^2) \| n^{-1} \|_{H^1}^2.
\]

Therefore
\[
\| n^{-1} \nabla n \|_{L^4} \lesssim (1 + \| k \|_{L^6}^{3/8}) \| k \|_{L^4}^{1/4} \| n^{-1} \|_{H^1}^{3/4}.
\]

Combining this inequality with (3.10) and using Lemma 2.2 to bound \( \| k \|_{L^4}, \| k \|_{L^6} \) and \( \| Ric \|_{L^2} \), it yields
\[
\| \nabla^2 (n^{-1}) \|_{L^2} \lesssim \| n^{-1} \|_{H^1}^{3/4} \| \nabla (n^{-1}) \|_{L^4} + (\| n^{-1} \|_{H^1} + 1) \| n^{-1} \|_{H^1} + \| \nabla (n^{-1}) \|_{L^4}.
\]

Applying Lemma 2.3 to the term \( \| \nabla (n^{-1}) \|_{L^4} \) gives
\[
\| \nabla^2 (n^{-1}) \|_{L^2} \lesssim \| n^{-1} \|_{H^1}^{3/4} \left( \| \nabla^2 (n^{-1}) \|_{L^2}^{3/4} \| \nabla (n^{-1}) \|_{L^2}^{1/4} + \| \nabla (n^{-1}) \|_{L^2} \right) + \| n^{-1} \|_{H^1}^2
\]
\[
+ \| \nabla^2 (n^{-1}) \|_{L^2}^{3/4} \| \nabla (n^{-1}) \|_{L^2}^{1/4} + \| n^{-1} \|_{H^1}^2.
\]

With the help of Young’s inequality, we obtain
\[
(3.12) \quad \| \nabla^2 (n^{-1}) \|_{L^2} \lesssim \| n^{-1} \|_{H^1} + \| n^{-1} \|_{H^1}.
\]

In order to estimate \( \| n^{-1} \|_{H^1} \), we use (3.11) with \( l = 3 \) to obtain
\[
\int_{\Sigma_t} (3 |\nabla (n^{-1})|^2 + n^{-3}) = \int_{\Sigma_t} |k|^2 n^{-2}.
\]
It then follows from the H"older inequality and Lemma 2.3 that
\[ \|\nabla(n^{-1})\|_{L^2} \lesssim \|k\|_{L^4} \|n^{-1}\|_{L^4} \lesssim \|k\|_{L^4} \left( \|\nabla(n^{-1})\|_{L^2}^{3/4} \|n^{-1}\|_{L^2}^{1/4} + \|n^{-1}\|_{L^2} \right). \]

This clearly implies
\[ \|\nabla(n^{-1})\|_{L^2} \lesssim (\|k\|_{L^4} + \|k\|_{L^4}^4) \|n^{-1}\|_{L^2} \lesssim \|n^{-1}\|_{L^2}. \]

The combination of (3.12) and (3.13) gives
\[ \|\nabla^2(n^{-1})\|_{L^2} + \|\nabla(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{L^2}^2 + \|n^{-1}\|_{L^2}. \]

Note that (3.11) with \( l = 2 \) gives
\[ \|n^{-1}\|_{L^2}^2 \leq \int_{\Sigma_t} \partial_t \|n^{-1}\|_{L^2}^2 \leq (\|k\|_{L^4}^2 - \|n^{-1}\|_{L^2}^2). \]

This implies \( \|n^{-1}\|_{L^2} \leq \|k\|_{L^4}^2 \leq C \). We therefore obtain \( \|n^{-1}\|_{H^2} \leq C \). With the help of Lemma 2.4, the estimate \( \|n^{-1}\|_{L^\infty} \leq C \) follows immediately. \( \Box \)

3.3. Derivative estimates about \( \dot{n} \). In this subsection we will give various estimates on the derivative \( \dot{n} := \partial_t \dot{n} \). We start with deriving an elliptic equation for \( \dot{n} \). By straightforward calculation we have
\[ \Delta \dot{n} = -\dot{g}^{ij} \nabla_i \nabla_j n + \partial_t (\Delta n) + \dot{g}^{ij} \dot{\Gamma}^a_{ij} \nabla_a n \]

Recall that
\[ \dot{\Gamma}^a_{ij} = \frac{1}{2} g^{ab} (\nabla_i \dot{g}_{jb} + \nabla_j \dot{g}_{ib} - \nabla_k \dot{g}_{ij}). \]

From (1.12), (1.16) and the fact Tr\( k = t \) it then follows
\[ \dot{g}^{ij} \dot{\Gamma}^a_{ij} \nabla_a n = -2k^a \nabla^i n \nabla_a n + \text{Trk}[\nabla n]^2. \]

Plugging this identity into (3.14) and using \( \dot{g}^{ij} = 2nk^{ij} \) and \( \Delta n = |k|^2 - 1 \) we obtain
\[ \Delta \dot{n} = -2nk^{ij} \nabla_i \nabla_j n + |k|^2 \dot{n} + \partial_t (|k|^2) - 2k^a \nabla^i n \nabla_a n + \text{Trk}[\nabla n]^2 \]

We may use the equations (1.2) and (1.3) to derive
\[ \partial_t (|k|^2) = -2k^a \nabla_i \nabla_j n + 2nR_{ij} k^{ij} + 2n |k|^2 \text{Trk}. \]

Consequently, we obtain
\[ \Delta \dot{n} = -4nk^{ij} \nabla_i \nabla_j n + |k|^2 \dot{n} - 2k^a \nabla^i n \nabla_a n + \text{Trk}[\nabla n]^2 \]
\[ + 2nR_{ij} k^{ij} + 2n |k|^2 \text{Trk}. \]

(3.15)

Now we multiply the equation (3.15) by \( \dot{n} \) and integrate over \( \Sigma_t \), by using the boundedness of \( n \) and the H"older inequality we obtain
\[ \int_{\Sigma_t} (|\nabla \dot{n}|^2 + |k|^2 |\dot{n}|^2) \]
\[ \lesssim \int_{\Sigma_t} (|\dot{n}| |\nabla^2 n| + |\dot{n}| |\nabla n|^2 + |\dot{n}| |\text{Ric}| |k| + |\dot{n}| |k|^3) \]
\[ \leq (\|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^4}^2 + \|\text{Ric}\|_{L^2}) \|k\|_{L^4} \|\dot{n}\|_{L^4} + \|k\|_{L^4}^2 \|\dot{n}\|_{L^2}. \]

Using the bounds derived in Lemma 2.2 and Proposition 3 together with the Sobolev embedding we have
\[ \int_{\Sigma_t} (|\nabla \dot{n}|^2 + |k|^2 |\dot{n}|^2) \lesssim \|\dot{n}\|_{L^4} + \|\dot{n}\|_{L^2} \lesssim \|\nabla \dot{n}\|_{L^2} + \|\dot{n}\|_{L^2}. \]
Recall that $|k|^2 = |\dot{k}|^2 + t^2/3$ and $|t| \geq |t_*| > 0$. Therefore
\[ \|\nabla \dot{n}\|_{L^2}^2 + \|\ddot{n}\|_{L^2}^2 \lesssim \|\nabla \dot{n}\|_{L^2}^2 + \|\ddot{n}\|_{L^2}. \]
We therefore obtain

**Lemma 3.2.** Let the assumption (A1) hold. Then for each $\Sigma_i \subset M_*$, there holds (3.16)
\[ \|\nabla \dot{n}\|_{L^2(\Sigma_i)} + \|\ddot{n}\|_{L^2(\Sigma_i)} \leq C. \]

Now we are ready to give some estimates on the mixed norms of $\dot{n}$.

**Proposition 7.** Let the assumption (A1) hold. Let $\dot{n} = \partial_t n$. Then there hold
\[ \|\nabla^2 \dot{n}\|_{L^2(M_*)} \leq C \quad \text{and} \quad \|\ddot{n}\|_{L^6(\mathcal{M}_*)} \leq C \]
for any $1 \leq b < 2$.

**Proof.** In view of the assumption (A1), it suffice to establish on every $\Sigma_i$ the inequalities
\[ \|\nabla^2 \ddot{n}\|_{L^2(\Sigma_i)} \lesssim C \left( ||k||_{L^\infty(\Sigma_i)} + 1 \right), \]
and
\[ \|\ddot{n}\|_{L^\infty(\Sigma_i)} \leq C \left( ||k||_{L^\infty(\Sigma_i)}^{3/2-3/p} + 1 \right) \]
for any $3 < p \leq 6$.

By the Böchner identity, we have
\[ \|\nabla^2 \ddot{n}\|_{L^2}^2 \leq \|\Delta \ddot{n}\|_{L^2}^2 + \|Ric\|_{L^2}\|\nabla \dot{n}\|_{L^2}^2. \]
By using $\|Ric\|_{L^2} \lesssim 1$ and applying Lemma 2.3 to $\|\nabla \dot{n}\|_{L^1}$, we obtain
\[ \|\nabla^2 \ddot{n}\|_{L^2} \lesssim \|\Delta \ddot{n}\|_{L^2} + \|\nabla \ddot{n}\|_{L^2}^{3/4}\|\nabla \dot{n}\|_{L^2}^{1/4} + \|\nabla \dot{n}\|_{L^2}. \]
In view of Young’s inequality and (3.16), it follows (3.19)
\[ \|\nabla^2 \ddot{n}\|_{L^2} \lesssim \|\Delta \ddot{n}\|_{L^2} + 1. \]
From the equation (3.14) it follows that
\[ \|\Delta \ddot{n}\|_{L^2} \lesssim \|k\|_{L^\infty} \left( ||\nabla^2 n\|_{L^2} + ||Ric\|_{L^2} \right) + ||k||_{L^\infty}^2||\dot{n}\|_{L^2} + \|\nabla \dot{n}\|_{L^6}||k||_{L^6} + ||k||_{L^6}^3. \]
With the help of the estimates derived in Lemma 2.2, Proposition 3 and (3.10) together with the Sobolev embedding we have $\|\Delta \ddot{n}\|_{L^2} \lesssim ||k||_{L^\infty} + 1$. Therefore $\|\nabla^2 \ddot{n}\|_{L^2} \lesssim ||k||_{L^\infty} + 1$ which is exactly (3.17). The inequality (3.18) immediately follows from Lemma 2.4 (3.17) and (3.10).

4. **Null radius of injectivity: proof of main theorem II**

In this section we will give the sketch of the proof of Theorem 1.2. The complete proof is rather involved and requires a delicate bootstrap argument. For any $t_0 < t_1 < t_*$ we consider the slab $M_I = \bigcup_{\ell \in I} \Sigma_i$ with $I = [t_0, t_1]$. We set, for each $p \in M_I$,
\[ \tilde{i}_*(p, t) = \begin{cases} +\infty, & \text{if } i_*(p, t) > (t(p) - t_0), \\ i_*(p, t), & \text{otherwise} \end{cases} \]
and define
\[ i_* := \min \{ \tilde{i}_*(p, t) : p \in M_I \}. \]
Due to the compactness of $M_I$, we have $i_* > 0$. In order to complete the proof of Theorem 1.2 it suffices to show that $i_* > \delta_*$ for some universal constant $\delta_* > 0$. 

We will use the following result concerning the lower bound on the null radius of injectivity of a globally hyperbolic space-time which has essentially been proved in \[11\].

**Theorem 4.1.** Let \( C^{-1} \leq n \leq C \) on \( \mathcal{M}_I \) for some constant \( C > 0 \). Then there exists a small constant \( \epsilon > 0 \) depending only on \( C \) such that if, for some constant \( \delta_* > 0 \), the following three conditions hold for all \( p \in \mathcal{M}_I \):

**C1.** the null radius of conjugacy satisfies

\[
s_*(p,t) > \min\{i_*,\delta_*\};
\]

**C2.** for each \( t \) satisfying

\[
0 \leq t(p) - t \leq \min\{i_*,\delta_*\},
\]

the metric \( \gamma_t \) on \( S^2 \), obtained by restricting the metric \( g \) on \( S_t \) to \( S_t := N^-(p) \cap \Sigma_t \) and then pulling it back to \( S^2 \) by the exponential map \( G(t,\cdot) \), verifies

\[
|\gamma_t(X,X) - \hat{\gamma}(X,X)| < \epsilon \hat{\gamma}(X,X), \quad \forall X \in T\Sigma^2,
\]

where \( \hat{\gamma} \) is the standard metric on \( S^2 \);

**C3.** On \( U_p := I_p \times B_{\delta_*}(p) \) with \( I_p := [t(p) - \min\{i_*,\delta_*\}, t(p)] \) and \( B_{\delta_*}(p) \subset \Sigma_{t(p)} \) a geodesic ball, there is a system of coordinates \( x^\alpha \) with \( x^0 = t \) relative to which the metric \( g \) is close to the Minkowski metric \( m_{\alpha\beta} = -n(p)dt^2 + \delta_{ij}dx^idx^j \) in the sense that

\[
|n - n(p)| + |g_{ij} - \delta_{ij}| < \epsilon \quad \text{on } U_p,
\]

then there holds \( i_*>\delta_* \), i.e. the null radius of injectivity verifies

\[
 i_*(p,t) > \min\{\delta_*,t(p) - t_0\}
\]

for all \( p \in \mathcal{M}_I \).

Let us briefly outline the idea of proof. Assume that \( i_* \leq \delta_* \). Let \( p_0 \in \mathcal{M}_I \) be a point such that \( i_*(p_0, t) = i_* \leq t(p) - t_0 \). By C1 we have \( s_*(p_0, t) > i_*(p_0, t) = l_*(p_0, t) \). Thus there exist two distinct past null geodesics \( \gamma_1 \) and \( \gamma_2 \) initiating at \( p_0 \) intersect at a point \( q_0 \) with \( t(q_0) = t(p_0) - i_* \). According to the definition of \( i_* \) and \( [11] \) Lemma 3.1 \( \gamma_1 \) and \( \gamma_2 \) are opposite at both \( p_0 \) and \( q_0 \). On the other hand, under the conditions C2 and C3, Lemma 3.2 and Lemma 3.3 in \[11\] imply that such two null geodesics can not intersect in the time slab \([t(p_0) - i_*, t(p_0)]\).

Theorem \[11\] provides a general framework to estimate the null radius of injectivity from below. Under the condition \([A1]\), in \[11\] Klainerman and Rodnianski showed that the conditions C1–C3 hold with a universal constant \( \delta_* > 0 \); thus they derived a universal lower bound on the null radius of injectivity.

In the following we will describe how to verify the conditions C1–C3 under the assumption (A1). To this end, for each \( p \in \mathcal{M}_I \) consider the past null cone \( N^-(p) \), let \( s \) be its affine parameter and let \( S_t = N^-(p) \cap \Sigma_t \). Then \( S_t \) is isometric to \( S^2 \) for each \( t \) satisfying \( t(p) - t_0 < t < t(p) \). Let \( \gamma \) be the restriction of \( g \) to \( S_t \) and let \( |S_t| \) be the corresponding area. The radius of \( S_t \) is defined to be

\[
r := \sqrt{(4\pi)^{-1}|S_t|}
\]

which is a function of \( t \) only.
On \(\mathcal{N}^-(p, \tau) \setminus \{p\}\) with \(\tau < i_*(p, t)\) we can define a conjugate null vector \(L\) with \(g(L, L) = -2\) and such that \(L\) is orthogonal to the leaves \(S_t\). In addition we can choose \((e_A)_{A=1,2}\) tangent to \(S_t\) such that \((e_A)_{A=1,2}, e_3 = L, e_4 = L\) form a null frame, i.e.

\[ g(L, L) = -2, \quad g(L, L) = g(L, e_A) = g(L, e_A) = 0, \quad g(e_A, e_B) = \delta_{AB}. \]

The null second fundamental forms \(\chi, \hat{\chi}\), the torsion \(\zeta\) and the Ricci coefficient \(\xi\) of the foliation \(S_t\) are then defined as follows

\[ \chi_{AB} = g(D_A L, e_B), \quad \chi^A_B = g(D_A L, e_B), \]

\[ \zeta_A = \frac{1}{2} g(D_A L, L), \quad \hat{\zeta}_A = \frac{1}{2} g(e_A, D_L L). \]

In addition we define

\[ \text{tr} \chi = \gamma^{AB} \chi_{AB}, \quad \hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} \text{tr} \gamma \chi_{AB}. \]

We can define \(\text{tr} \chi\) and \(\hat{\chi}\) similarly.

We introduce the null lapse function

\[ a^{-1} := g(L, T). \]

Then \(a > 0\) and \(a(p) = 1\). It is easy to see that

\[ L = -a^{-1}(T + N), \quad L = -a(T - N), \]

where \(N\) denotes the unit inward normal to \(S_t\) in \(\Sigma_t\). We also introduce the function

\[ \nu := -n^{-1} \nabla n + k_{NN} \]

which is relevant to the estimate on \(a\).

For any \(S_t\)-tangent tensor field \(F\) we define the norm \(|F|_{L^2(N^-(p, \tau))}\) by

\[ \|F\|_{L^2(N^-(p, \tau))} := \sup_{\omega \in \mathbb{S}^2} \int_{t(p) - \tau}^{t(p)} |F|^2 n\, dt = \sup_{\omega \in \mathbb{S}^2} \int_{\Gamma_\omega} |F|^2 n\, dt, \]

where \(\Gamma_\omega\) denotes the portion of a past null geodesic initiating from \(p\) contained in \(N^-(p, \tau)\).

The following result is sufficient to prove the conditions C1–C3 in Theorem 4.1.

**Theorem 4.2.** Let the assumption (A1) hold. Then there exist universal constants \(\delta_1 > 0\) and \(C_1 > 0\) such that for any \(p \in M_1\) there hold

\[ \int_{t(p) - \tau}^{t(p)} |k(\Phi(t))|^2 \, dt \leq C_1 \]

with \(\Phi\) the integral curve of \(T\) through \(p\), and

\[ |a - 1| \leq \frac{1}{2}, \quad \left| \text{tr} \chi - \frac{2}{s} \right| \leq C_1, \quad \left\| \hat{\chi} \right\|^2_{L^2(N^-(p, \tau))} \leq C_1 \]

on any null cones \(N^-(p, \tau)\), where \(\tau := \min\{i_*, \delta_1\}\).

In fact, the estimate on \(\text{tr} \chi\) in (4.4) implies the condition C1, see [6, 11]. Next we will show that the estimates in (4.4) imply the condition C2. To see this, we recall that \(\frac{d}{dt} = -na\) and \(\frac{d}{ds} \gamma_{AB} = 2 \gamma_{AB}\). Then

\[ \frac{d}{dt} (s^{-2} \gamma_{AB}) = -na \left( -2s^{-3} \gamma_{AB} + 2s^{-2} \chi_{AB} \right) \]
Let \( X \in T^3 \) be any vector field. We integrate the above equation along any null geodesic and note that \( \lim_{t \to t(p)} s(t)^{-2} \gamma(t) = \gamma(t) \) (see [13]), it follows that
\[
\left| s(t)^{-2} \gamma(X, X) - \tilde{\gamma}(X, X) \right| \leq \int_t^{t(p)} \left( 2|\tilde{\chi}| + \left| \text{tr} \chi - \frac{2}{s} \right| \right) s(t')^{-2} \gamma(X, X) \, ndt'
\]
Let
\[
\Theta := 2|\tilde{\chi}| + \left| \text{tr} \chi - \frac{2}{s} \right|.
\]
We then have
\[
\left| s(t)^{-2} \gamma(X, X) - \tilde{\gamma}(X, X) \right| \leq \int_t^{t(p)} \Theta \left| s(t')^{-2} \gamma(X, X) - \tilde{\gamma}(X, X) \right| \, ndt'
\]
\[
+ \tilde{\gamma}(X, X) \int_t^{t(p)} \Theta(t') \, ndt'.
\]
Therefore, it follows from the Gronwall inequality that
\[
\left| s(t)^{-2} \gamma(X, X) - \tilde{\gamma}(X, X) \right| \leq \tilde{\gamma}(X, X) \int_t^{t(p)} \Theta(t') \, ndt' \exp \left( \int_t^{t(p)} \Theta(t') \, ndt' \right).
\]

Since \( 0 < n \leq 3/t^2 \), the estimate (4.3) in Theorem 4.2 implies
\[
\int_t^{t(p)} \Theta \, ndt' \leq C \left( (t(p) - t)^{1/2} + (t(p) - t) \right) \leq C(t(p) - t)^{1/2}
\]
and consequently
\[
(4.5) \quad \left| s^{-2} \gamma(X, X) - \tilde{\gamma}(X, X) \right| \leq C(t(p) - t)^{1/2} \tilde{\gamma}(X, X)
\]
for all \( t(p) - \min\{i_*, \delta_*\} \leq t < t(p) \), where \( C \) is a universal constant. The condition C2 is thus verified.

The verification of the condition C3, using the estimate (4.3), is given in the following result.

**Lemma 4.3.** Let the assumption (A1) hold. For any \( \epsilon > 0 \), there exists a constant \( \delta_0 > 0 \) depending only on \( Q_0, K_0, t_* \) and \( \epsilon \) such that for every point \( p \in M \), there exists an \( U_p := I_p \times B_{\delta_0}(p) \) with \( I_p = [t(p) - \min\{i_*, \delta_*\}, t(p)] \) a system of transported coordinates \( t, x = (x^1, x^2, x^3) \) relative to which \( g \) is close to the Minkowski metric \( m(p) = -n(p)^2 dt^2 + \delta_{ij} dx^i dx^j \), in the sense that
\[
|g_{ij} - \delta_{ij}| < \epsilon \quad \text{and} \quad |n - n(p)| < \epsilon.
\]

**Proof.** It follows from Proposition 2 that there exists a constant \( \delta_0 > 0 \) depending only \( K_0, Q_0, t_* \) and \( \epsilon \) such that every geodesic ball \( B_{\delta_0}(p) \subset \Sigma_{t(p)} \) admits a system of harmonic coordinates \( x = (x^1, x^2, x^3) \) under which
\[
(4.7) \quad (1 + \epsilon/2)^{-1} \delta_{ij} \leq g_{ij} \leq (1 + \epsilon/2) \delta_{ij}.
\]
Under the transported coordinates \( t, x = (x^1, x^2, x^3) \), let \( p = (t(p), 0) \) and let \( q = (t, x) \) be an arbitrary point in \( I_p \times B_{\delta_0}(p) \) with \( I_p = [t(p) - \min\{i_*, \delta_*\}, t(p)] \), where \( 0 < \delta_* \leq \delta_0 \) is a constant to be determined. By using the equation \( \partial_t g_{ij} = -2nk_{ij} \) we have
\[
|g_{ij}(t, x) - g_{ij}(t(p), x)| = \left| \int_t^{t(p)} \partial_t g_{ij}(t', x) \, dt' \right| = 2 \int_t^{t(p)} n|k| \, dt'.
\]
Using the bound $0 < n < 3/4$, the Hölder inequality and the estimate \((4.3)\) in Theorem 4.2, it follows for some universal constant $C_1 > 0$ that

$$|g_{ij}(t, x) - g_{ij}(t(p), x)| \leq C_1 (t(p) - t)^{1/2} \leq C_1 \delta_*^{1/2}.$$ 

In view of \((4.7)\), we thus obtain

\[(4.8) \quad |g_{ij}(t, x) - \delta_{ij}| \leq |g_{ij}(t, x) - g_{ij}(t(p), x)| + |g_{ij}(t(p), x) - \delta_{ij}| \leq C_1 \delta_*^{1/2} + \epsilon/2,\]

which gives the first inequality in \((4.6)\) by letting $C_1 \delta_*^{1/2} < \epsilon/2$.

Next we prove the second inequality in \((4.6)\). From Proposition 7 we have

$$|n(t, x) - n(t(p), x)| \leq \int_t^{t(p)} |\dot{n}(t', x)| dt' \leq (t(p) - t)^{1/4} \|\dot{n}\|_{L^4_t L^\infty_x} \leq C_2 \delta_*^{1/4},$$

while by employing Morrey’s estimate, Lemma 2.3 and Proposition 3 we have

$$|n(t(p), x) - n(t(p), 0)| \leq C_2 \delta_*^{1/4} \|\nabla n\|_{L^4(S_{t(p)})} \leq C_2 \delta_*^{1/4} (\|\nabla^2 n\|_{L^3_t L^2_x} + \|\nabla n\|_{L^2_x}) \leq C_2 \delta_*^{1/4},$$

where $C_2 > 0$ is a universal constant. Therefore

$$|n(t, x) - n(t(p), 0)| \leq 2C_2 \delta_*^{1/4}$$

which implies the second inequality in \((4.3)\) by further letting $2C_2 \delta_*^{1/4} < \epsilon$. \(\square\)

The proof of Theorem 4.2 is based on a delicate bootstrap argument. We first fix some notations and terminology. Related to the deformation tensor $\pi_{\alpha\beta}$ of $T$, we introduce the $\Sigma$-tangent tensor $h^\alpha_\alpha h^\beta_\beta \pi_{\mu\nu}$, where

$$h^\alpha_\alpha = \delta^\alpha_\alpha + T^\alpha_\alpha$$

denotes the projection tensor. It is easy to see that $k_{ij} = h^\alpha_\alpha h^\beta_\beta \pi_{\mu\nu}$ and thus this tensor is an extension of $k$. We will denote it by the same notation $k$, i.e.

\[(4.9) \quad k_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta \pi_{\mu\nu}\]

Note that $k_{0\alpha} = k_{\alpha 0} = 0$.

Corresponding to the null vector $L$, let $\nabla_L k$ be the $\Sigma$-tangent tensor defined by

$$\nabla_L k_{ij} := h^\alpha_\alpha h^\beta_\beta D_L k_{\alpha\beta}$$

and let

$$|\nabla_L k|^2 = g^{ij} g^{ij'} \nabla_L k_{ij} \nabla_L k_{ij'}. $$

We also introduce $\nabla k$ by $\nabla_A k_{ij} := \nabla_A k_{ij}$ and set

$$|\nabla k|^2 = g^{AB} g^{ij} \nabla_A k_{ij} \nabla_B k_{ij'}. $$

Corresponding to the second fundamental form $k$, then, for each $p \in M_1$, we introduce on the null cone $N^-(p, \tau)$ the $k$-flux

\[(4.10) \quad \mathcal{F}[k](p, \tau) = \int_{N^-(p, \tau)} (|\nabla k|^2 + |\nabla_L k|^2), \]

where, for each function $f$ and $\tau < i_*(p, t)$,

$$\int_{N^-(p, \tau)} f := \int_{i_*(p, t)}^{t(p)} \int_{S_t} f n a d \mu_s dt. $$
Corresponding to the time foliation, we recall the null components of the Riemannian curvature tensor $R$ as follows
\begin{align}
\alpha_{AB} &= R(L, e_A, L, e_B), \\
\beta_A &= \frac{1}{2} R(e_A, L, L, L), \\
\rho &= \frac{1}{4} R(L, L, L, L), \\
\sigma &= \frac{1}{4} R(L, L, L, L), \\
\beta_A &= \frac{1}{2} R(e_A, L, L, L), \\
\underline{a}_{AB} &= R(L, e_A, L, e_B).
\end{align}

The corresponding curvature flux $R(p, \tau)$ on the null cone $\mathcal{N}^-(p, \tau)$ is given by
\begin{align*}
R(p, \tau) &= \int_{t(p) - \tau}^{t(p)} \int_{S_t} (|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\beta|^2) \, \nu d\mu_p dt.
\end{align*}

The following result says that once the null lapse $a$ is well controlled, then the $k$-flux and the curvature flux can be bounded by a universal constant.

**Theorem 4.4.** Let the condition (A1) hold. Then there exists a universal constant $C_*>1$ such that for all $p \in M_I$ if $|a - 1| \leq 1/2$ on $\mathcal{N}^-(p, \tau)$ for some $0 < \tau \leq i_*$ then there holds
\begin{align*}
R(p, \tau) + F[k](p, \tau) \leq C_*.
\end{align*}

We will prove Theorem 4.4 in Section 6. This result requires $1/2 \leq a \leq 3/2$ on $\mathcal{N}^-(p, \tau)$ which is obvious for small $\tau > 0$ since $a(p) = 1$. In order for the above result to be applicable, we must show that there is a universal constant $\delta_* > 0$ such that the same bound on $a$ holds with $\tau := \min\{i_*, \delta_*\}$, and so does the same bound on $R(p, \tau) + F[k](p, \tau)$. We will use a bootstrap argument to achieve this together with various estimates on $\text{tr}\chi$, $\dot{\chi}$ and $\nu$. That is, we will make the following bootstrap assumptions
\begin{align}
\text{(BA1)} & \quad |a - 1| \leq \frac{1}{2}, \\
\text{(BA2)} & \quad \left| \text{tr}\chi - \frac{2}{s} \right| \leq \mathcal{E}_0, \\
\text{(BA3)} & \quad \|\dot{\chi}\|_{L^2_t L^2_x(\mathcal{N}^-(p, \tau))} \leq \mathcal{E}_0, \\
\text{(BA4)} & \quad \|\nu\|_{L^2_t L^2_x(\mathcal{N}^-(p, \tau))} \leq \mathcal{E}_0,
\end{align}
on the null cone $\mathcal{N}^-(p, \tau)$ for all $p \in M_I$, where $0 < \tau \leq i_*$ and $\mathcal{E}_0 \geq 1$ are two numbers satisfying $\mathcal{E}_0 \tau \leq 1$. Due to the continuity of the quantities involved and the compactness of $M_I$, the bootstrap assumptions (BA1)-(BA4) hold automatically for sufficiently small $\tau > 0$. Our goal is to show that we can choose universal constants $\mathcal{E}_0 \geq 1$ and $\delta_* > 0$ such that (BA1)-(BA4) hold with $\tau = \min\{i_*, \delta_*\}$. We will achieve this by showing that the estimates in (BA1)-(BA4) can be improved.

We will first derive various intermediate consequences of the bootstrap assumptions. In particular, we will derive the estimate on the important quantity $\mathcal{N}_1[f]$ which is defined as follows. For any $S_t$ tangent tensor field $F$ defined on the null cone $\mathcal{N}^-(p, \tau)$, the Sobolev norm $\mathcal{N}_1[F](p, \tau)$ is defined by
\begin{align}
\mathcal{N}_1[F](p, \tau) := \|F^{-1}F\|_{L^2(\mathcal{N}^-(p, \tau))} + \|
abla L F\|_{L^2(\mathcal{N}^-(p, \tau))} + \|\nabla F\|_{L^2(\mathcal{N}^-(p, \tau))}.
\end{align}
Recall that the components of the deformation tensor $\pi$ of $\mathbf{T}$ under transported coordinates are given by $\pi_{00} = 0$, $\pi_{0i} = -n^{-1}\nabla_i n$ and $\pi_{ij} = k_{ij}$. Let us denote by
\[ \lambda = -\frac{1}{3} \text{Tr} k = -\frac{1}{3} t \text{ and } \hat{k} \text{ the traceless part of } k. \] We decompose \( \hat{k} \) on each \( S_t \) by introducing components

\[ \eta_{AB} = \hat{k}_{AB}, \quad \epsilon_A = \hat{k}_{AN}, \quad \delta = \hat{k}_{NN} \]

where \( (e_A)_{A=1,2} \) is an orthonormal frame on \( S_t \) and \( N \) is the inward unit normal of \( S_t \) in \( \Sigma_t \). Let \( \hat{\eta}_{AB} \) be the traceless part of \( \eta \). Since \( \delta^{AB} \hat{\eta}_{AB} = -\delta \), we have

\[ \hat{\eta}_{AB} = \eta_{AB} + \frac{1}{2} \delta_{AB} \delta. \]

We will denote by \( \hat{k}, \nabla \hat{k} \) and \( \pi_0 \) the collections

\[ \hat{k} = (\delta, \epsilon, \hat{\eta}), \quad \nabla \hat{k} = (\nabla \delta, \nabla \epsilon, \nabla \hat{\eta}), \quad \pi_0 = (\nabla \log n, \nabla N \log n) \]

respectively. We then define \( \pi \) to be the collection

\[ \pi = (\hat{k}, \pi_0, \lambda). \]

We define \( N_1[\pi](p, \tau) \) according to (4.12) with \( F \) replaced by \( \pi \).

With the help of the bound on \( k \)-flux given in Theorem 4.4 and various estimates on the lapse \( n \) given in Section 3, we will show that \( N_1[\pi](p, \tau) \) can be bounded in a suitable way under (A1) and the bootstrap assumptions.

\textbf{Theorem 4.5.} Let (A1) hold. Then there exists a universal constant \( C \) such that under the bootstrap assumptions (BA1)–(BA3) with \( E_0 \tau \leq 1 \) there holds

\[ N_1[\pi](p, \tau) \leq C \]

for all \( p \in \mathcal{M}_I \).

We will prove Theorem 4.5 in Section 8. From Theorem 4.4 and Theorem 4.5 it follows that

\[ \mathcal{R}(p, \tau) + N_1[\pi](p, \tau) \leq C_0, \]

where \( C_0 \geq 1 \) is a universal constant.

With the help of (4.16), we can establish the following result which enables us to improve the estimates in the bootstrap assumptions.

\textbf{Theorem 4.6.} There exist two universal constants \( \delta_0 > 0 \) and \( C_1 \geq 1 \) such that, under the bootstrap assumptions (BA1)–(BA4) with \( E_0 \tau \leq 1 \), if \( \tau < \min\{i_*, \delta_0\} \) then there hold

\[ |a - 1| \leq C_1 \tau^{1/2}, \]

\[ |\text{tr} \chi - \frac{2}{s}| \leq C_1, \]

\[ \|\hat{\chi}\|_{L^2_t L^2(N^-(p, \tau))} \leq C_1, \]

\[ \|\nu\|_{L^2_t L^2(N^-(p, \tau))} \leq C_1 \]

on the null cones \( \mathcal{N}^-(p, \tau) \) for all \( p \in \mathcal{M}_I \).

The significance of Theorem 4.6 lies in that it allows us to choose \( E_0 \geq 1 \) and \( \delta_* > 0 \) universal such that (BA1)–(BA4) hold on \( \mathcal{N}^-(p, \tau) \) with \( \tau = \min\{i_*, \delta_*\} \). To see this, we choose \( E_0 \) and \( \delta_* \) in the way that

\[ E_0 := 2C_1 \quad \text{and} \quad \delta_* = \min\{(4C_1)^{-2}, \delta_0\}. \]
With such $\mathcal{E}_0$ and $\delta_*$, the estimates (4.17)–(4.20) imply that the estimates (BA1)–(BA4) can be improved as
\[
|a - 1| \leq \frac{1}{4}, \quad \tr \chi - \frac{2}{s} \leq \frac{1}{2} \mathcal{E}_0, \quad \|\chi\|^2_{L^\infty_t L^2(N^-(p,\tau))} \leq \frac{1}{2} \mathcal{E}_0, \quad \|\nu\|^2_{L^2_t L^\infty_t(N^-(p,\tau))} \leq \frac{1}{2} \mathcal{E}_0
\]
on $N^-(p,\tau)$ if $\tau \leq \min\{i_*, \delta_*\}$. By repeated use of Theorem 4.4 Theorem 4.5 and Theorem 4.6 the bootstrap principle implies that the estimates in the bootstrap assumptions (BA1)–(BA4) hold with $\tau = \min\{i_*, \delta_*\}$, where $\mathcal{E}_0$ and $\delta_*$ are determined by (4.21) which are positive universal constants. Consequently, we obtain (4.3) in Theorem 4.2.

We remark that the analogous results to Theorem 4.6 have been proved in [7, 13] for the geodesic foliations where only the bound of the curvature flux is used. In time foliations, however, the proof of Theorem 4.6 relies not only on the curvature flux but also on $N_1[\gamma]$.

Assuming (4.20), the following simple argument shows how to derive (4.17) with the help of (BA1). Recall that $a^{-1} = g(L, T)$ and $L = -a^{-1}(N + T)$. We have
\[
\frac{d}{ds} a^{-1} = g(L, DL T) = a^{-2} g(N, DT T) + a^{-2} g(N, DN T).
\]
Since $DT T = n^{-1} \nabla n$ and $k_{NN} = -\langle N, DN T \rangle$ we obtain $\frac{d}{ds} a^{-1} = -a^{-2} (\pi_0 N + k_{NN})$. Consequently
\[
(4.22) \quad L(a) = \frac{d}{ds} a = \pi_0 N + k_{NN}.
\]
Since $\frac{d}{dt} a = -na$, we have
\[
\frac{d}{dt} a = -na (\pi_0 N + k_{NN}).
\]
Integrating the above equation along null geodesics initiating from $p$ and using $a(p) = 1$ yields
\[
a - 1 = \int^t_t (\pi_0 N + k_{NN}) n dt' = \int^t_t n m dt'.
\]
Since $0 < n \leq 3/l_*^2$, (BA1) and (4.20) imply
\[
|a - 1| \leq C_1 (t(p) - t)^{1/2} \leq C_1 \tau^{1/2}
\]
for all $t(p) - \tau < t < t(p)$, where $C_1$ could be a different but universal constant.

The derivation of (4.18)–(4.20) however is highly nontrivial and requires lengthy calculation. The complete proof is contained in [14] where other related estimates for Ricci coefficients are proved simultaneously.

In order to complete the proof of Theorem 4.2 it remains to prove (4.3) which is restated in the following result.

**Theorem 4.7.** Assume that the condition (A1) holds. Then there exist universal constants $\delta_* > 0$ and $C > 0$ such that
\[
\int_{t(p) - \min\{i_*, \delta_*\}}^{t(p)} |k(\Phi(t))|^2 n dt \leq C
\]
for all $p \in \mathcal{M}_I$, where $\Phi$ denotes the integral curve of $T$ through $p$. 
The proof of Theorem 1.7 forms the core part of the present paper. It is based on the formula of $\Box k$ given in Section 5 and a Kirchoff-Sobolev representation for $k$ given in Section 9 together with various estimates on null cones derived in Section 8.

5. Tensorial wave equation for the second fundamental form

In this section we will derive the formula for $\Box k$, where $k$ is defined in \[1.9\] whose projection to $\Sigma_t$ is exactly the second fundamental form.

**Proposition 8.** The tensor $k$ defined by \[4.9\] verifies the tensorial wave equation

\[\Box k_{ij} = -n^{-3}n_i\nabla_i n_j + n^{-2}\nabla_i\nabla_j n + 2\pi_0a \left(\nabla^a k_{ij} - \nabla_i k^a_j - \nabla_j k^a_i\right)\]

\[-2Trk_k_{ij} - Rk_{ij} + RTrk g_{ij} + 2(k_a^a R_{aj} + k_j^a R_{ai}) - 2R_{ab}k_{ab} g_{ij}\]

\[+ n^{-1}(2k^a_i \nabla_a \nabla_j n + 2k^a_j \nabla_a \nabla_i n + \Delta nk_{ij} - Trk_n \nabla \nabla j n)\]

\[+ 2k_{ia}k^a_{jb} - \pi_0a\pi_0^a k_{ij} - n^{-1}k_{ij}.\]  \(\text{(5.1)}\)

**Proof.** We first recall that

\[\Box k_{ij} = -D_0D_0k_{ij} + g^{pq}D_pD_qk_{ij}.\]

By using $k_{0a} = k_{a0} = 0$ and $D_1e_j = \nabla_1e_j - k_{ij} T$, we can obtain through a straightforward calculation that

\[g^{pq}D_pD_qk_{ij} = \Delta k_{ij} + Trk D_0k_{ij} + 2k_{ia}k^a_{jb}.\]

By using $D.T = n^{-1}\nabla_i e_i = -\pi_0^a e_i$ and $k_{0a} = k_{a0} = 0$, we can obtain

\[D_0D_0k_{ij} = e_0(D_0k_{ij}) + k^a_0D_0k_{aj} + k^a_jD_0k_{ia} + \pi_0a \nabla^a k_{ij}\]

\[+ \pi_0aD_0k_{0j} + \pi_0aD_0k_{0i}.\]

It is easy to see $D_0k_{0j} = \pi_0a k^a_j$. From the equation \[1.3\] it also follows that

\[D_0k_{ij} = e_0(k_{ij}) + 2k_{ia}k^a_j = -n^{-1}\nabla_i\nabla_j + R_{ij} + Trk k_{ij}.\]  \(\text{(5.2)}\)

Consequently

\[D_0D_0k_{ij} = e_0(D_0k_{ij}) + \pi_0a \nabla^a k_{ij} - n^{-1}(k^a_i \nabla_a \nabla_j n + k^a_j \nabla_a \nabla_i n)\]

\[+ (k^a_i R_{aj} + k^a_j R_{ai}) + 2Trk k_{ia}k^a_j + \pi_0a \pi_0a k^a_j + \pi_0a \pi_0a k^a_i.\]

Therefore

\[\Box k_{ij} = -e_0(D_0k_{ij}) - \pi_0a \nabla^a k_{ij} - \pi_0a \pi_0a k^a_j - \pi_0a \pi_0a k^a_i\]

\[+ n^{-1}(k^a_i \nabla_a \nabla_j n + k^a_j \nabla_a \nabla_i n) - (k^a_i R_{aj} + k^a_j R_{ai})\]

\[+ 2k_{ia}k^a_j + \Delta k_{ij} + Trk D_0k_{ij} + 2k_{ia}k^a_{jb}.\]  \(\text{(5.3)}\)

We need to compute $e_0(D_0k_{ij})$. It follows from \[5.2\] and $Trk = t$, we have

\[e_0(D_0k_{ij}) = n^{-1}\nabla_i\nabla_j n - n^{-2}\partial_t(\nabla_i \nabla j n) + n^{-1}\partial_t R_{ij}\]

\[+ n^{-1}k_{ij} + Trk D_0k_{ij} - 2Trk k_{ia}k^a_j.\]  \(\text{(5.4)}\)

In order to compute $\partial_t(\nabla_i \nabla_j n)$ and $\partial_t R_{ij}$, let $\Gamma^a_{ij}$ denote the Christoffel symbol of $\Sigma_t$. Then it follows from the equation $\partial_ig_{ij} = -2nk_{ij}$ that

\[\Gamma^a_{ij} = -n(\nabla^a k^a_{ij} + \nabla_j k^a_i - \nabla^a k_{ij}) - \nabla_i nk^a_j - \nabla_j nk^a_i + \nabla^a nk_{ij}.\]
Using $\text{div}k = 0$ and $\text{Tr}k = t$, this in particular implies $\dot{\Gamma}^i_{aj} = -\text{Tr}k \nabla_j n$. Therefore, noting that $\partial_i(\nabla_i \nabla_j n) = \nabla_i \nabla_j \dot{n} - \Gamma^a_{ij} \nabla_a n$, we can obtain

$$\partial_i(\nabla_i \nabla_j n) = \nabla_i \nabla_j \dot{n} + n \nabla_i n (\nabla_i k^a_j + \nabla_j k^a_i - \nabla^a k_{ij})$$

(5.5)

+ $$(\nabla_i n k^a_j + \nabla_j n k^a_i) \nabla_a n - |\nabla n|^2 k_{ij}.$$ 

Noting also that $\partial_i R_{ij} = \nabla_a \Gamma^a_{ij} - \nabla_i \Gamma^a_{aj}$ and $\text{div} k = 0$, we have

$$\partial_i R_{ij} = \nabla_a n (2\nabla^a k_{ij} - \nabla_i k^a_j - \nabla_j k^a_i) - n (\nabla_a \nabla_i k^a_j + \nabla_a \nabla_j k^a_i - \triangle k_{ij})$$

and

$$\nabla_a \nabla_i k^a_j = [\nabla_a, \nabla_i] k^a_j = R^a_{bik} k^b + R_{aij} k^a_j$$

and the curvature decomposition formula

$$R^a_{bik} = 2R_{ia} k^a_j + R_{ja} k^a_i - \text{Tr} R_{ij} k^a + R_{ab} k^{ab} g_{ij} - \frac{1}{2} (R_{ij} k^a + R_{ki} n^a) R_n.$$

we obtain

$$\nabla_a \nabla_i k^a_j = 2R_{ia} k^a_j + R_{ja} k^a_i - \text{Tr} R_{ij} k^a + R_{ab} k^{ab} g_{ij} - \frac{1}{2} (R_{ij} k^a + R_{ki} n^a) R_n.$$

Consequently

$$\partial_i R_{ij} = \nabla_a n (2\nabla^a k_{ij} - \nabla_i k^a_j - \nabla_j k^a_i) - (\nabla_a \nabla_i n k^a_j + \nabla_a \nabla_j n k^a_i)$$

+ $n \triangle k_{ij} + n \triangle n k_{ij} - 3 n (R_{ia} k^a_j + R_{ja} k^a_i) + 2 n \text{Tr} R_{ij} R_n$

(5.6)

$$+ 3 n R_{ab} k^{ab} g_{ij} + 3 n R_{ij} - n \text{Tr} R_{ij} R_n + \text{Tr} \nabla_i \nabla_j n.$$

Plugging (5.5) and (5.6) into (5.4), and using $\pi_0 = -n^{-1} \nabla_i n$, it yields

$$e_0(D_0 k_{ij}) = n^{-3} \nabla_i \nabla_j n - n^{-2} \nabla_i \nabla_j \dot{n} - \pi_0 (3\nabla^a k_{ij} - 2\nabla_i k^a_j - 2\nabla_j k^a_i) - \pi_0 \pi_0 k_{ij}$$

- $\pi_0 \pi_0 k_{ij} - \pi_0 \pi_0 k_{ij} - \pi_0 \pi_0 k_{ij} - \pi_0 \pi_0 k_{ij} - \pi_0 \pi_0 k_{ij}$

+ $\triangle k_{ij} + n^{-1} \triangle n k_{ij} - 3 (R_{ia} k^a_j + R_{ja} k^a_i) + 2 \text{Tr} R_{ij} R_n + 2 \text{Tr} R_{ab} k^{ab} g_{ij}$

+ $R_{ij} - n \text{Tr} R_{ij} R_n + \text{Tr} D_0 k_{ij} - 2 \text{Tr} k^a k_{aj}.$

Plugging the above equation into (5.5) gives the desired equation. \(\square\)

6. Proof of Theorem 4.4

In this section we will complete the proof of Theorem 4.4 i.e. we will show that if $|a - 1| \leq 1/2$ on $\mathcal{N}^-(p, \tau)$ for some $0 < \tau \leq \tau_*$ then

$$\mathcal{R}(p, \tau) + \mathcal{F}[k](p, \tau) \leq C_*,$$

where $C_*$ is a universal constant.

We will use the following result (see [3, Lemma 8.1.1]).

**Lemma 6.1.** Let $P$ be a vector field defined on the domain $\mathcal{J}^-(p, \tau)$. Then

$$\int_{\mathcal{N}^-(p, \tau)} g(P, L) = \int_{\mathcal{J}^-(p, \tau)} D^\mu P_\mu - \int_{\Sigma_{\tau(p)} - \mathcal{J}^-(p)} g(P,T) d\mu_g,$$
Thus, there holds, for some universal constant $C > Q \pi$ as

\[
\int_{J^-(p, \tau)} f = \int_{t(p) - \tau}^{t(p)} dt \left( \int_{\Sigma_t \cap J^-(p)} f nd\mu_g \right).
\]

We first show the boundedness of the curvature flux $\mathcal{R}(p, \tau)$. With the Bel-Robinson tensor $Q[R]$ defined in Section 2, we introduce $P_\mu = Q[R]_{\mu\beta\gamma\delta} T^\beta T^\gamma T^\delta$. We may apply Lemma 6.1 to obtain

\[
\int_{N^-(p, \tau)} g(P, L) = \int_{J^-(p, \tau)} D^\mu P_\mu - \int_{\Sigma_{t(p) - \tau} \cap J^-(p)} Q[R](T, T, T) d\mu_g,
\]

With the help of the calculations in subsection 2.1, (A1) and Lemma 2.1, it then yields

(6.1) \[
\left| \int_{N^-(p, \tau)} g(P, L) \right| \lesssim C.
\]

Note that $g(P, L) = Q[R](T, T, T, T)$ and $T = -\frac{1}{2}(aL + a^{-1} L)$. Since $|a - 1| \leq 1/2$ on $N^-(p, \tau)$, it follows from [3] Lemma 7.3.1 that $-g(P, L)$ is equivalent to

\[
|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\sigma|^2.
\]

Thus, there holds, for some universal constant $C > 0$,

\[
C^{-1} \mathcal{R}(p, \tau) \leq \left| \int_{N^-(p, \tau)} g(P, L) \right| \leq C \mathcal{R}(p, \tau).
\]

By (6.1), we conclude that $\mathcal{R}(p, \tau) \leq C_*$ for some universal constant $C_*$. Next we will show the boundedness of the $k$-flux $\mathcal{F}[k](p, \tau)$. With the help of the projection tensor

\[
h^{\alpha\beta} = g^{\alpha\beta} + T^\alpha T^\beta,
\]

for any tensor field $U_{\alpha_1\alpha_2\ldots\alpha_m}$ in $TM$, we define $|U|$ as follows

\[
|U|^2 = h^{IJ} U_I U_J = h^{\alpha_1\beta_1} \ldots h^{\alpha_m\beta_m} U_{\alpha_1\alpha_2\ldots\alpha_m} U_{\beta_1\beta_2\ldots\beta_m}
\]

\[
h^{IJ} = h^{\alpha_1\beta_1} \ldots h^{\alpha_m\beta_m}, \quad U_I = U_{\alpha_1\alpha_2\ldots\alpha_m}, \quad U_J = U_{\beta_1\beta_2\ldots\beta_m}.
\]

For any $\Sigma_t$-tangent tensor field $U$ in $M_I$, we define the energy momentum tensor $Q[U]_{\alpha\beta}$ associated with the covariant wave operator acting on tensors:

\[
Q[U]_{\alpha\beta} := h^{IJ} D_\alpha U_I D_\beta U_J - \frac{1}{2} g_{\alpha\beta} h^{IJ} g^{\mu\nu} D_\mu U_I D_\nu U_J.
\]

We have

\[
D^\beta Q[U]_{\alpha\beta} = h^{IJ} D_\alpha U_I \Box U_J + h^{IJ}(D_\beta D_\alpha U_I - D_\alpha D_\beta U_I) D^\beta U_J
\]

\[
+ D^\beta h^{IJ} (D_\alpha U_I D_\beta U_J - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} D_\mu U_I D_\nu U_J)
\]

It is easy to see that the last term in the above equation can be written symbolically as $\pi \cdot DU \cdot DU$. 

\[\]
Now we apply the above equation to \( U = k \). Noting that \( h^{0\alpha} = 0 \) and \( h^{ij} = g^{ij} \), we have

\[
D^\beta(Q[k]_{\alpha\beta} T^\alpha) = D^\beta T^\alpha Q[k]_{\alpha\beta} + D^\beta Q[k]_{\alpha\beta} \\
= -k^{ij} Q[k]_{ij} - \pi^{0j} Q[k]_{0j} + D_0 k^{ij} \Box k_{ij} \\
+ [D_a, D_0] k_{ij} \nabla^a k^{ij} + \pi \cdot D k \cdot D k.
\]

(6.2)

In view of the commutation formula

\[
[D_m, D_0] k_{ij} = R^b_{ij} m_0 k_{bj} + R^b_{jm} m_0 k_{ib} = -\epsilon^{ij}_{km} H_{sm} k^b_j - \epsilon^{ij}_{km} H_{sm} k^b_i,
\]

we derive symbolically

\[
D^\beta(Q[k]_{\alpha\beta} T^\alpha) = -k^{ij} Q[k]_{ij} - \pi^{0j} Q[k]_{0j} + D_0 k^{ij} \Box k_{ij} \\
+ H \cdot k \cdot \nabla k + \pi \cdot D k \cdot D k.
\]

(6.3)

From the definition of \( Q[k] \), it is easy to see that

\[
Q[k]_{00} = \frac{1}{2} (|D_0 k|^2 + |\nabla k|^2),
\]

(6.4)

\[
Q[k]_{0j} = D_0 k_{pq} \nabla_j k^{pq},
\]

(6.5)

\[
Q[k]_{ij} = \nabla_i k_{pq} \nabla_j k^{pq} - \frac{1}{2} \delta_{ij} (|D_0 k|^2 + |\nabla k|^2).
\]

(6.6)

Therefore

\[
D^\beta(Q[k]_{\alpha\beta} T^\alpha) = \frac{1}{2} \text{Tr} k (-|D_0 k|^2 + |\nabla k|^2) + k \cdot \nabla k \cdot \nabla k \\
+ D_0 k \cdot \Box k + H \cdot k \cdot \nabla k + \pi \cdot D k \cdot D k.
\]

(6.7)

We now apply Lemma 6.1 to \( P^\alpha := T^\alpha Q[k]_{\alpha}^{\beta} \) and obtain

\[
\int_{N^-(p, \tau)} Q[k](T, L) + \int_{\Sigma_{(p, \tau)} \cap J^-(p)} Q[k]_{00} = \int_{J^+(p, \tau)} D^\beta(Q[k]_{\alpha\beta} T^\alpha).
\]

(6.8)

For the null pair \( L \) and \( L \), it is easy to see that

\[
Q[k](L, L) = |\nabla L k|^2, \quad Q[k](\Sigma, L) = |\nabla k|^2.
\]

Since \( T = -\frac{1}{2}(aL + a^{-1} L) \), we have

\[
Q[k](T, L) = -\frac{1}{2} (aQ[k](L, L) + a^{-1} Q[k](\Sigma, L)) = -\frac{1}{2} (a|\nabla L k|^2 + a^{-1} |\nabla k|^2).
\]

Since \( |a - 1| \leq 1/2 \), the \( k \)-flux defined in (4.10) verifies the inequality

\[
- \int_{N^-(p, \tau)} Q[k](T, L) \leq F[k](p, \tau) \leq -4 \int_{N^-(p, \tau)} Q[k](T, L).
\]

Thus we derive from (6.8) and (6.4) that

\[
F[k](p, \tau) \leq 4 \left[ \int_{J^- (p, \tau)} D^\beta(Q[k]_{\alpha\beta} T^\alpha) \right] + 2 \int_{\Sigma_{(p, \tau)} \cap J^- (p)} (|D_0 k|^2 + |\nabla k|^2).
\]

(6.9)

In view of (5.2), Lemma 2.1, Proposition 3 and Proposition 4, we have

\[
\int_{\Sigma_+} (|D_0 k|^2 + |\nabla k|^2) \lesssim \|\nabla^2 n\|^2_{L^2(\Sigma_t)} + \|Ric\|^2_{L^2(\Sigma_t)} + \|k\|^4_{L^4(\Sigma_t)} + \|\nabla k\|^2_{L^2(\Sigma_t)} \leq C.
\]

(6.10)
Moreover, in view of (6.7), (A1), Lemma 2.2 and the above inequality we have
\[
\left| \int_{\mathcal{F}^{-}(p, \tau)} \mathbf{D}^3(Q[k]_{\alpha\beta} T^\alpha) \right| \leq \int_{t(p)-\tau}^{t(p)} \| \mathbf{D}_3 k \|_{L^2(\Sigma_t)} \| \Box k \|_{L^2(\Sigma_t)} \, dt' \\
+ \int_{t(p)-\tau}^{t(p)} \| \pi \|_{L^\infty(\Sigma_t)} \left( \| \mathbf{D}_3 k \|_{L^2(\Sigma_t)}^2 + \| \nabla k \|_{L^2(\Sigma_t)}^2 \right) \, dt' \\
+ \int_{t(p)-\tau}^{t(p)} \| k \|_{L^\infty(\Sigma_t)} \| H \|_{L^2(\Sigma_t)} \| \nabla k \|_{L^2(\Sigma_t)} \, dt' \\
\leq \int_{t(p)-\tau}^{t(p)} \| \Box k \|_{L^2(\Sigma_t)} \, dt' + \int_{t(p)-\tau}^{t(p)} \| \pi \|_{L^\infty(\Sigma_t)} \, dt' \\
\leq C + C \int_{t(p)-\tau}^{t(p)} \| \Box k \|_{L^2(\Sigma_t)} \, dt'.
\]

Therefore
\[
\mathcal{F}[k](p, \tau) \leq C + C \int_{t(p)-\tau}^{t(p)} \| \Box k \|_{L^2(\Sigma_t)} \, dt'.
\]

We now recall the formula for \( \Box k \) given in Proposition 8 which symbolically can be written as
\[
\Box k = -n^{-3} \tilde{n} \nabla^2 n + n^{-2} \nabla^2 \tilde{n} + \pi \cdot \pi + k \cdot \nabla^2 n + k \cdot \nabla \tilde{n} - n^{-1} k.
\]

Since \( C^{-1} \leq n \leq C \), we obtain
\[
\| \Box k \|_{L^1_t L^2_x} \lesssim \|
\nabla^2 n \|_{L^1_t L^2_x} + \|
\nabla^2 \tilde{n} \|_{L^1_t L^2_x} + \|
\pi \|_{L^1_t L^\infty_x} \| \pi \|_{L^2_t L^2_x}^2 \\
+ \|
\pi \|_{L^1_t L^\infty_x} \| \nabla^2 n \|_{L^1_t L^\infty_x} + \|
\pi \|_{L^1_t L^\infty_x} \| \tilde{n} \|_{L^1_t L^\infty_x} \| \nabla \tilde{n} \|_{L^1_t L^2_x} \\
+ \|
\pi \|_{L^1_t L^\infty_x} + \|
\pi \|_{L^1_t L^\infty_x} \| \nabla k \|_{L^1_t L^2_x}.
\]

In view of the assumption (A1), Lemma 2.2, Proposition 3, Proposition 7 and (6.10), it follows
\[
\| \Box k \|_{L^1_t L^2_x} \leq C \left( 1 + \| \pi \|_{L^1_t L^\infty_x} + \| \tau \right) \leq C.
\]

Combining the above inequality with (6.12) completes the proof of Theorem 4.4.

7. Trace estimates

For a point \( p \in \mathcal{M} \), let \( s \) be the affine parameter on the null cone \( \mathcal{N}^-(p) \) and let \( r \) be the radius of \( S_r := \mathcal{N}^-(p) \cap \Sigma_r \) which is defined by (4.2). On each \( S_r \) we introduce the ratio of area elements
\[
(7.1) \quad v(t) = \frac{\sqrt{\gamma}}{\sqrt{\gamma}}, \quad \omega \in \mathbb{S}^2.
\]

We will first show that all the quantities \( s, r, v^1_1 \) and \( t(p) - t \) are comparable under the bootstrap assumptions (BA1)–(BA3). Here we say two quantities \( \varphi \) and \( \psi \) are comparable in the sense that \( C^{-1} \psi \leq \varphi \leq C \psi \) for some universal constant \( C > 0 \).

Lemma 7.1. Under the bootstrap assumptions (BA1)–(BA3), the four quantities \( s(t), r(t), v^1_1 \) and \( t(p) - t \) are comparable on the null cone \( \mathcal{N}^-(p, \tau) \) with \( \tau \leq \min\{i, \delta \} \), where \( \delta > 0 \) is a universal constant.
Proof. The comparability of \( s \) and \( t(p) - t \) follows from the relation \( \frac{ds}{dt} = -na \) and the bootstrap assumption (BA1). Similar to the derivation of (1.3), we have under the bootstrap assumptions (BA1)–(BA3) that

\[
(7.2) \quad |\gamma - s(t)^2| \leq \frac{1}{2}s(t)^2
\]

for all \( t(p) - \min\{i_\ast, \tau, \delta_\ast\} \leq t < t(p) \), where \( \delta_\ast \) is a universal constant. This implies immediately that \( \frac{1}{2}s(t)^2 \leq v_t \leq \frac{3}{2}s(t)^2 \). Consequently \( v_t \) and \( t(p) - t \) are comparable. Thus for the area \( |S_t| \) of \( S_t \) there holds

\[
C^{-1}(t(p) - t)^2 \leq |S_t| \leq C(t(p) - t)^2
\]

for some universal constant \( C \). This together with the definition of \( r \) gives the comparability of \( r \) and \( t(p) - t \). \( \square \)

7.1. Optical function. In this section we give a brief review of the construction of optical functions, one may see [5] for more information.

For any point \( p \in M_1 \), let \( J^-(p) \) be the causal past and let \( N^-(p) \) and \( I^-(p) \) denote respectively the null boundary and the interior. For each \( 0 < \tau < i_\ast \) with \( i_\ast \) defined by (4.1), let \( J^-(p, \tau), N^-(p, \tau) \) and \( I^-(p, \tau) \) denote the portions of \( J^-(p) \), \( N^-(p) \) and \( I^-(p) \) in the time slab \([t(p) - \tau, t(p)]\) respectively. Let \( \Phi \) be the integral curve of \( T \) through \( p \) with \( \Phi(t(p)) = p \). According to the definition of \( i_\ast \), all the null cones \( N^-(\Phi(t), \tau + t - t(p)) \) with \( t(p) - \tau \leq t \leq t(p) \) and \( \tau < i_\ast \), are disjoint and their union forms \( N^-(p, \tau) \). We now define \( u \) to be the function, constant on each \( N^-(\Phi(t), t + \tau - t(p)) \), such that

\[
u(\Phi(t)) = \int_{t_0}^t n(\Phi(t'))dt'.
\]

Such \( u \), which will be called an optical function, is a well-defined smooth function on \( J^-(p, \tau) \) and satisfies the eikonal equation

\[
g^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0.
\]

It is clear that the level sets \( C_u \) of \( u \) are the incoming null cones in the time slab \([t(p) - \tau, t(p)]\) with vertices on \( \Phi \), and \( T(u) = 1 \) on \( \Phi \). Moreover, the null geodesic vector \( L \) defined before can be written as

\[
L = g^{\alpha\beta}\partial_\beta u\partial_\alpha.
\]

For each \( t \in [t(p) - \tau, t(p)] \), we define \( u_M(t) \) and \( u_m(t) \) respectively to be the largest and smallest values of \( u \) for which the part of the cone \( C_u \) that lies in the future of \( \Sigma_t \) is contained in \( J^-(p) \), i.e.

\[
u_M(t) = u(p) \quad \text{and} \quad u_m(t) = u(\Phi(t)).
\]

For each \( u(\Phi(t(p) - \tau)) \leq u \leq u(p) \), we also define \( t_M(u) \) and \( t_m(u) \) to be the largest and smallest value of \( t \) for which \( \Sigma_t \) intersects \( C_u \) respectively. It is clear that \( t_M(u) \) is the value of \( t \) at the vertex of \( C_u \) and \( t_m(u) = t(p) - \tau \). Note that both \( u_M \) and \( u_m \) are independent of \( t \).

We set

\[
S_{t,u} := C_u \cap \Sigma_t
\]

which is a smooth surface for each \( t(p) - \tau \leq t \leq t(p) \) and \( u_M \leq u < u_m(t) \). The corresponding radius function is defined as

\[
r(t, u) := \sqrt{(4\pi)^{-1}|S_{t,u}|},
\]
where \(|S_{t,u}|\) denotes the area of \(S_{t,u}\) with respect to the metric \(\gamma\).

The following result follows immediately from Lemma 7.1 and the definition of \(u\).

**Proposition 9.** Under the bootstrap assumptions (BA1)–(BA3) on \(N^-(p, \tau)\) for all \(p \in M_I\), there hold

\[(7.3) \quad C^{-1} \leq \frac{t_M(u) - t}{r(t, u)} \leq C\]

and

\[(7.4) \quad C^{-1} \leq \frac{u - u_m(t)}{r(t, u)} \leq C\]

for all \(t(p) - \min\{i_*, \tau, \delta_*\} < t < t(p)\), where \(C\) and \(\delta_*\) are two positive universal constants.

In view of the above notations, it is clear that

\[N^-(p, \tau) = \bigcup_{t \in [t(p) - \tau, t(p)]} S_{t,u} M.\]

Let \(\text{Int}(S_{t,u} M)\) be the interior of \(S_{t,u} M\) in \(\Sigma_t\), then

\[\text{Int}(S_{t,u} M) = \bigcup_{u \in [u_M, u_{m}(t)]} S_{t,u} \quad \text{and} \quad J^-(p, \tau) = \bigcup_{t \in [t(p) - \tau, t(p)]} \text{Int}(S_{t,u} M).\]

The following simple result can be found in [5].

**Lemma 7.2.** For any scalar \(f\) satisfying

\[\lim_{u \to u_{m}(t)} \int_{S_{t,u}} f d\mu_\gamma = 0,\]

there holds

\[\int_{S_{t,u} M} f d\mu_\gamma = -\int_{u_{m}(t)}^{u_M} \int_{S_{t,u}} (\nabla N f + t \theta f) ad\mu_\gamma u du,\]

where \(N\) denotes the unit inward normal to \(S_{t,u}\) in \(\Sigma_t\), and \(\theta\) denotes the corresponding second fundamental form.

### 7.2. Trace estimates

We will rely on the following trace inequality.

**Lemma 7.3** (Trace inequality). Under the bootstrap assumptions (BA1)–(BA3) on \(N^-(p, \tau)\) with \(E_{0,\tau} \leq 1\), for any \(\Sigma_t\) tangent tensor field \(F\) there holds

\[\|r^{-1/2} F\|_{L^2(S_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)},\]

where \(S_t := N^-(p, \tau) \cap \Sigma_t\) and \(r := \sqrt{(4\pi)^{-1}|S_t|}\).

The proof of Lemma 7.3 can be seen in Appendix. Using Lemma 7.3, we are able to derive the following

**Proposition 10.** Let the bootstrap assumptions (BA1)–(BA3) hold on \(N^-(p, \tau)\) with \(E_{0,\tau} \leq 1\). Then for any \(\Sigma_t\) tangent tensor field \(F\) there hold

\[(7.5) \quad \|F\|_{L^2(S_t)} \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)},\]

\[(7.6) \quad \|F\|_{L^4(S_t)} \lesssim \|F\|_{H^1(\Sigma_t)}\]

for all \(t(p) - \tau \leq t < t(p)\).
Proof. Let \( \phi(u) \) be a smooth cut-off function verifying \( 0 \leq \phi \leq 1 \), \( \phi(u_M) = 1 \) and \( \text{supp}(\phi) \subset \left[ \frac{u_m + u_M}{2}, u_M \right] \). It then follows from Lemma 7.2 that

\[
(7.7) \quad \|F\|_{L^2(S_t)}^2 = -\int_{\text{Int}(S_t)} \left( \nabla_N [\phi F]^2 + \text{tr} \theta |\phi F|^2 \right) d\mu_\gamma du' = I_1 + I_2,
\]

where

\[
I_1 = -2 \int_{\text{Int}(S_t)} \left( \phi^2 F \cdot \nabla_N F + \phi \nabla_N |\phi F|^2 \right) d\mu_\gamma du',
\]

\[
I_2 = -\int_{\text{Int}(S_t)} \text{tr} \theta |\phi F|^2 d\mu_\gamma du'.
\]

Since the bootstrap assumption (BA1) implies \( 1/2 \leq a \leq 3/2 \), it is easy to see that

\[
\left| \int_{\text{Int}(S_t)} \phi^2 F \cdot \nabla_N F d\mu_\gamma du' \right| \lesssim \|\nabla_N F\|_{L^2(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}
\]

and

\[
\left| \int_{\text{Int}(S_t)} \phi \nabla_N |\phi F|^2 d\mu_\gamma du' \right| \lesssim \frac{1}{u_M - u_m} \int_{\frac{u_m + u_M}{2}}^{u_M} \int_{S_t, u'} |F|^2 d\mu_\gamma du'.
\]

It follows from Lemma 7.3 that

\[
\int_{S_t, u'} |F|^2 d\mu_\gamma \lesssim \|\nabla F\|_{L^2(S_t, u')} \|F\|_{L^2(S_t, u')} r^{1/2}
\]

\[
\lesssim \|F\|_{H^1(S_t)} \|F\|_{L^2(S_t, u')} r^{1/2},
\]

where \( r := r(t, u') \). From Proposition 9 it follows that \( r(t, u') \lesssim u' - u_m \). Thus

\[
\left| \int_{\text{Int}(S_t)} \phi \nabla_N |\phi F|^2 d\mu_\gamma du' \right|
\]

\[
\lesssim \frac{1}{u_M - u_m} \|F\|_{H^1(S_t)} \|F\|_{L^2(\Sigma_t)} \left( \int_{\frac{u_m + u_M}{2}}^{u_M} (u' - u_m) du' \right)^{1/2}
\]

\[
\lesssim \|F\|_{H^1(S_t)} \|F\|_{L^2(\Sigma_t)}.
\]

We therefore obtain

\[
|I_1| \lesssim \|F\|_{H^1(S_t)} \|F\|_{L^2(\Sigma_t)}.
\]

In order to estimate the term \( I_2 \), we recall that \( \text{tr} \theta = -a \text{tr} \chi + \delta AB k_{AB} \). Since the bootstrap assumption (BA2) implies \( |\text{tr} \chi - 2/s| \leq \mathcal{E}_0 \); on each \( S_t, u' \) and Proposition 9 implies that \( s, t(p) - t \) and \( r \) are comparable, we have

\[
|I_2| \lesssim (\mathcal{E}_0 + 1) \int_{u_m}^{u_M} \int_{S_t, u'} r^{-1} |\phi F|^2 d\mu_\gamma du' + \int_{u_m}^{u_M} \int_{S_t, u'} |k| |\phi F|^2 d\mu_\gamma du'
\]

\[
\lesssim \int_{u_m}^{u_M} \int_{S_t, u'} r^{-1} |\phi F|^2 d\mu_\gamma du' + \|k\|_{L^3(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}^2.
\]

Recall that \( \|k\|_{L^3(\Sigma_t)} \leq C \) from Lemma 2.2 and apply Lemma 2.3 to \( \|F\|_{L^2(\Sigma_t)}^2 \) we obtain

\[
|I_2| \lesssim \|F\|_{H^1(S_t)} \|F\|_{L^2(\Sigma_t)} + \int_{\frac{u_m + u_M}{2}}^{u_M} \int_{S_t, u'} r^{-1} |F|^2 d\mu_\gamma du'.
\]
Now we use Lemma 7.3 again and note that Proposition 9 implies $r(t, u')^{-1} \lesssim (u' - u_m)^{-1}$, we have

$$\int_{t_0}^{t_1} \int_{S_1, u'} r^{-1} |F|^2 d\mu_{\gamma} du' \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} \left( \int_{t_0}^{t_1} (u' - u_m)^{-1} du' \right)^{1/2} \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$ 

Therefore

$$|I_2| \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$ 

The proof of (7.5) is complete.

Applying (7.7) with $|F|$ replaced by $|F|^2$, combined with Sobolev embedding, we can obtain (7.6) in the similar fashion. $\square$

As a consequence, we obtain

**Proposition 11.** Let the bootstrap assumptions (BA1)-(BA3) hold on $\mathcal{N}^{-}(p, \tau)$ with $\mathcal{E}_0 \tau \leq 1$. Let $S_t := \mathcal{N}^{-}(p, \tau) \cap \Sigma_t$ and let $r$ be defined by (4.2). Let $\pi_0$ denote the tensor $-\nabla \log n$.

(a) Let $\pi$ denote either $k$, $\pi_0$ or $D_0 \log n$, then for $t(p) - \tau \leq t \leq t(p)$

\[ \|\pi\|_{L^2(S_t)} \leq C, \tag{7.8} \]

\[ \|r^{-1/2} \pi\|_{L^2(S_t)} \leq C. \tag{7.9} \]

(b) Let $F$ denotes either $n^{-1} \nabla^2 n$ or $n^{-2} \nabla \dot{n}$, then

\[ \|F\|_{L^2(\mathcal{N}^{-}(p, \tau))} \leq C. \tag{7.10} \]

(c) For $\pi_0$, there holds

\[ \|\nabla_r \pi_0\|_{L^2(\mathcal{N}^{-}(p, \tau))} + \|D_0 \pi_0\|_{L^2(\mathcal{N}^{-}(p, \tau))} + \|\nabla \pi_0\|_{L^2(\mathcal{N}^{-}(p, \tau))} \leq C. \tag{7.11} \]

**Proof.** (a) From Lemma 2.2 Proposition 5 and Lemma 3.2 it follows that $\|\pi\|_{H^1(\Sigma_t)} \leq C$. Thus (7.8) follows from (4.9) in Proposition 10 and (7.9) follows from Lemma 7.3.

(b) For $F = (n^{-1} \nabla^2 n, n^{-2} \nabla \dot{n})$ it follows from Proposition 3 Proposition 4 Lemma 3.2 and Proposition 7 that

\[ \|\nabla F\|_{L^1_t L^2(\mathcal{M}_s)} \leq C \quad \text{and} \quad \|F\|_{L^1_t L^2(\mathcal{M}_s)} \leq C. \]

Applying (7.5) to $F$ yields

\[ \|F\|_{L^2(\mathcal{N}^{-}(p, \tau))} \lesssim \|F\|_{L^1_t H^1(\mathcal{M}_s)} \|F\|_{L^1_t L^2(\mathcal{M}_s)} \lesssim C. \]

(c) By straightforward calculation, symbolically we have

\[ D_0 \pi_0 = -n^{-2} \nabla \dot{n} + \pi \cdot \pi_0, \]

\[ \nabla \pi_0 = -n^{-1} \nabla^2 n + \pi \cdot \pi_0, \]

\[ \nabla_r \pi_0 = a^{-1} n^{-2} \nabla \dot{n} - a^{-1} \nabla \pi_0 - a^{-1} \pi \cdot \pi_0. \]

Therefore, (7.11) follows immediately from (7.8) and (7.10). $\square$
8. Estimates on the null cones

8.1. Structure equations on the null cones. In Section 4 we introduced the null pair \( L, L' \) on the null cone \( \mathcal{N}^-(p, \tau) \) and define the null second fundamental forms \( \chi, \chi' \) and the Ricci coefficients \( \zeta, \zeta' \). For the null frame \((e_A)_{A=1,2}, e_3 = L, e_4 = L'\), there hold

\[
\begin{align*}
D_AL &= \chi_{AB}e_B + \zeta_A L, & D_AL &= \chi_{AB}e_B - \zeta_A L, \\
D_L L &= 2\zeta_A e_A + 2\omega L, & D_L L &= 2\zeta_A e_A - 2\omega L, \\
D_L L &= 2\zeta_A e_A, & D_L L &= 0
\end{align*}
\]

(8.1)

and

\[
\begin{align*}
D_B e_A &= \nabla_B e_A + \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\chi_{AB} e_4, \\
D_4 e_A &= \nabla_4 e_A + \zeta_A e_4, \\
D_3 e_A &= \nabla_3 e_A + \zeta_A e_3 + \xi_A e_4,
\end{align*}
\]

(8.2)

where \( \nabla \) denotes the covariant differentiation on \( S_t \).

Let \( \alpha, \beta, \rho \) and \( \sigma \) be the null components of \( R \) defined in (4.11). There hold the following structure equations on null cones (see [5, p.351–360].)

\[
\begin{align*}
\frac{d}{ds} \text{tr} \chi &= \frac{1}{2}(\text{tr} \chi)^2 = -|\hat{\chi}|^2, \\
\frac{d}{ds} \hat{\chi}_{AB} &= \text{tr} \chi \hat{\chi}_{AB} = \alpha_{AB}, \\
\frac{d}{ds} \zeta_A &= -\chi_{AB} \zeta_B + \chi_{AB} \zeta'_{AB} - \beta_A, \\
\frac{d}{ds} \text{div} \chi &= 2 \text{div} \zeta - \hat{\chi} \cdot \hat{\chi} + 2|\zeta|^2 + 2\rho.
\end{align*}
\]

Moreover, \( \zeta \) verifies the following Hodge system

\[
\begin{align*}
\text{div} \zeta &= -\mu - \rho + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - |\zeta|^2 - \frac{1}{2} a \delta \text{tr} \chi - a \lambda \text{tr} \chi, \\
\text{curl} \zeta &= \sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\chi},
\end{align*}
\]

where \( \mu \) and \( \mu' \) are the mass aspect functions defined by

\[
\begin{align*}
\mu &= -\frac{1}{2} D_3 \text{tr} \chi + \frac{a^2}{4}(\text{tr} \chi)^2 - \omega \text{tr} \chi, \\
\mu' &= D_3 \text{tr} \chi + \frac{1}{2} \text{tr} \chi \cdot \text{tr} \chi, \\
\omega &= \frac{1}{2}(D_3 \log a + ak_N - a\pi_0 N).
\end{align*}
\]

Let \( N \) be the unit inward normal to \( S_t \) in \( \Sigma_t \) and let \( \theta \) be the second fundamental form of \( S_t \), i.e. \( \theta_{AB} = g(\nabla_A N, e_B) \). Then there hold

\[
\begin{align*}
\nabla_N e_A &= \nabla_N e_A + a^{-1} \nabla_A a N, \\
\nabla_A N &= \theta_{AB} e_B, \\
\nabla_B e_A &= \nabla_B e_A - \theta_{AB} N, \\
\nabla_N N &= -a^{-1} \nabla_A a e_A.
\end{align*}
\]
We introduce the new null pair $L' := T + N$, $L := T - N$. Then $L = -a^{-1}L'$ and $\bar{L} = -aL'$. Let $\chi'$ be the Ricci coefficients corresponding to the null frame $(e_A)_{A=1,2}$, $e_3 = L'$, $e_4 = L$. Then

$$\chi = -a^{-1}\chi', \quad \bar{\chi} = -a\bar{\chi}'$$

and

\begin{align*}
\chi'_{AB} &= \theta_{AB} - k_{AB}, \\
\bar{\chi}'_{AB} &= -\theta_{AB} - k_{AB}, \\
\zeta'_A &= \nabla_A \log a + \epsilon_A, \\
\bar{\zeta}'_A &= \nabla_A \log n - \epsilon_A, \\
\nu &= -\nabla_N \log n + \delta - \lambda.
\end{align*}

8.2. Proof of Theorem 4.5 The main purpose of this subsection is to prove Theorem 4.5 concerning the boundedness of $N_1[F]$ under the bootstrap assumptions (BA1)--(BA3) on $N^-(p, \tau)$ with $0 < \tau \leq \tau_s$ and $E_0 \tau \leq 1$ for any $p \in M_I$, where $\hat{\pi}$ is defined by (4.14) and the Sobolev norm $N_1[F]$ for any $S_t$ tangent tensor field $F$ is defined by (4.12). We can restate Theorem 4.5 in the following form, since the estimates for $\lambda$ are trivial.

**Proposition 12.** Let $\hat{\pi}$ be the $S_t$ tangent tensor field defined in (4.14), and let $\bar{\pi} := (k, -\nabla \log n)$. Then, under the bootstrap assumptions (BA1)--(BA4) with $E_0 \tau \leq 1$, there hold

\begin{align*}
\|r^{-1}\bar{\pi}\| &\lesssim C, \\
\|\nabla \hat{\pi}\| &\lesssim C, \\
\|\bar{\nabla}_L \hat{\pi}\| &\lesssim C.
\end{align*}

We have obtained in Theorem 4.4 and 7.11 that

\begin{equation}
\|\nabla \bar{\pi}\|_{L^2(\mathbb{S}^2)} + \|\nabla L \bar{\pi}\|_{L^2(\mathbb{S}^2)} \leq C.
\end{equation}

In view of (8.14), (8.15) and (8.1), (8.3), we can symbolically write

\begin{equation}
\nabla \hat{\pi} = \nabla \bar{\pi} + \text{tr} \theta \cdot \hat{\pi} + \bar{\theta} \cdot \hat{\pi}
\end{equation}

and also in view of $\frac{d\bar{\pi}}{ds} = -(an)^{-1}$,

\begin{equation}
\nabla_L \bar{\pi} = \nabla_L \bar{\pi} + \hat{\pi} \cdot \zeta + (an)^{-1}.
\end{equation}

In order to show Proposition 12 we need three auxiliary lemmas. We will use the following norms for $S_t$ tangent tensor fields $F$ on null cones $N^-(p, \tau)$

\begin{align*}
\|F\|_{L^2_0(N^-(p, \tau))}^q &:= \int_{S^2 \cap \Gamma_\omega} \sup_{t \in \Gamma_\omega} \langle v_t | F | g \rangle^q d\mu_{g^2}, \\
\|F\|_{L^2_0(N^-(p, \tau))}^q &:= \int_{S^2 \cap \Gamma_\omega} |F|^q g d\mu_{g^2}.
\end{align*}

where $v_t$ is defined by (7.1), and $\Gamma_\omega \in S^2$, denotes the portion of an incoming null geodesic initiating from $p$ in the time slab $[t(p) - \tau, t(p)]$. In the following argument we will suppress $N^-(p, \tau)$ in these norms for simplicity.
Lemma 8.1. For any $S_t$ tangent tensor field $F$, there hold the estimates

\begin{equation}
\|r^{-1/2}F\|_{L^2_L L^\infty_t} + \|F\|_{L^1_L L^\infty_t} \lesssim N_1[F],
\end{equation}

\begin{equation}
\|F\|_{L^1_L L^\infty_t}^2 \lesssim (\|\nabla L F\|_{L^2} + \|r^{-1}F\|_{L^2}) \|F\|_{L^\infty L^2_t}.
\end{equation}

Proof. We refer to [7, 13] for the proof of (8.28). In the following we will prove (8.29). Let $v_t$ be defined by (7.1). We first integrate along any past null geodesic initiating from $p$ to get

\begin{equation}
v_t|F|^4 = \lim_{t \to t(p)} (v_t|F|^4) - \int_t^{t(p)} \frac{d}{dt} (v_t|F|^4) dt'.
\end{equation}

For the estimate of the first term on the right of (8.29), we proceed as follows.

Let $\varphi$ be a smooth cut-off function defined on $[t(p) - \tau, t(p)]$ verifying $0 \leq \varphi \leq 1$, $\varphi(t(p)) = 1$ and $\text{supp}\, \varphi \subset [t(p) - \tau, t(p)]$. Then

\begin{equation}
\lim_{t \to t(p)} (v_t|F|^4)^{t(p)-\tau} \left( \frac{d}{dt} (v_t|F|^4) + 4v_t|F|^4 \varphi \frac{d}{dt} \varphi \right) dt.
\end{equation}

Since $\left| \frac{d}{dt} \varphi \right| \lesssim (t(p) - t)^{-1}$, we have from Lemma (7.1) that $\left| \frac{d}{dt} \varphi \right| v_t^{t(p)-\tau} \lesssim 1$. Using $0 \leq \varphi \leq 1$, it then follows from (8.30) and (8.31) that

\begin{equation}
\|F\|_{L^1_L L^\infty_t}^2 = \int_{S^2} \sup_{t(p)-\tau \leq t \leq t(p)} (v_t|F|^4) \lesssim I + II,
\end{equation}

where

\begin{align*}
I = \int_{S^2} \int_{t(p)-\tau}^{t(p)} \left| \frac{d}{dt} (v_t|F|^4) \right| dt, \quad II = \int_{S^2} \int_{t(p)-\tau}^{t(p)} v_t^{t(p)-\tau} |F|^4.
\end{align*}

Since

\begin{equation}
\frac{d}{dt} (v_t|F|^4) = -na (\text{tr} \varphi v_t|F|^4 + 4v_t|F|^2 \nabla L F \cdot F),
\end{equation}

we have

\begin{align*}
I &\lesssim \left( \|v_t^{1/2} \nabla L F\|_{L^2_L L^\infty_t} + \|\text{tr} v_t^{1/2} F\|_{L^2_L L^\infty_t} \right) \|F\|_{L^\infty L^2_t} \|v_t^{1/2} F\|_{L^2 L^\infty_t} \\
&\lesssim (\|\nabla L F\|_{L^2} + \|\text{tr} F\|_{L^2}) \|F\|_{L^\infty L^2_t} \|F\|_{L^2_L L^\infty_t}.
\end{align*}

By the bootstrap assumption (BA2) and Lemma (7.1) we have

\begin{equation}
\|\text{tr} F\|_{L^2} \lesssim \|\nabla L F\|_{L^2} + \|r^{-1}F\|_{L^2} \lesssim (E_0 \tau + 1) \|r^{-1}F\|_{L^2} \lesssim \|r^{-1}F\|_{L^2}.
\end{equation}

Therefore

\begin{equation}
I \lesssim (\|\nabla L F\|_{L^2} + \|r^{-1}F\|_{L^2}) \|F\|_{L^\infty L^2_t} \|F\|_{L^2_L L^\infty_t}^2.
\end{equation}

It is easy to see that

\begin{align*}
|II| &\lesssim \|F\|_{L^2_L L^\infty_t} \|F\|_{L^\infty L^2_t} \|v_t^{1/2} F\|_{L^2_L L^\infty_t} \lesssim \|r^{-1}F\|_{L^2} \|F\|_{L^\infty L^2_t} \|F\|_{L^2_L L^\infty_t}^2.
\end{align*}

Combining the estimates for $I$ and $II$ with (8.32) gives (8.29).
Lemma 8.2. For any $S_t$ tangent tensor field $F$ verifying
\begin{equation}
\nabla^2 F + \frac{m}{2} \text{tr} \chi F = G \cdot F + H \tag{8.33}
\end{equation}
with $m \geq 1$ an integer and $G$ a tensor field of suitable type, if $\lim_{t \to t(p)} r(t)^m F = 0$ and $\sup_{\omega \in S^2} \int_{t(p) - \tau}^{t(p)} na|G|^2 dt \leq \Delta_0^2$, the following estimates hold
\begin{align*}
\|F\|_{L^2_t L^2_x} & \lesssim e^{C \Delta_0 \tau^{1/2}} \|H\|_{L^2}, \\
\|r^2 F\|_{L^2_t L^\infty_x} & \lesssim e^{C \Delta_0 \tau^{1/2}} \|H\|_{L^2}.
\end{align*}

Proof. In view of sup $\omega \in S^2 \int_{t(p) - \tau}^{t(p)} na|G|^2 dt \leq \Delta_0^2$, we have
\begin{equation}
\|F\|_{L^2_t L^2_x} \lesssim e^{C \Delta_0 \tau^{1/2}} \|H\|_{L^2},
\end{equation}
\begin{equation}
\|r^2 F\|_{L^2_t L^\infty_x} \lesssim e^{C \Delta_0 \tau^{1/2}} \|H\|_{L^2}.
\end{equation}

By a simple argument we can derive
\begin{equation}
v^m_t |F|^2 \leq \exp \left( \int_t^{t(p)} |G|na \right) \int_t^{t(p)} \text{na}|G|^2 dt \exp \left( - \int_t^{t(p)} na|G| \right) \text{d}t'.
\end{equation}

In view of sup $\omega \in S^2 \int_{t(p) - \tau}^{t(p)} na|G|^2 dt \leq \Delta_0^2$, we have $\exp(\int_t^{t(p)} na|G|) \leq e^{C \Delta_0 \tau^{1/2}}$. Thus by using Lemma 7.1 and $m \geq 1$, we have
\begin{equation}
|F| \leq e^{C \Delta_0 \tau^{1/2}} v_{t(p)}^{-m/2} \int_t^{t(p)} v_{t'}^{-m/2} |H| \text{nad} t' \lesssim e^{C \Delta_0 \tau^{1/2}} (t(p) - t)^{-1} \int_t^{t(p)} r |H| dt'.
\end{equation}

To derive (8.34), we integrate the above inequality along a null geodesic initiating from vertex $p$. By the Hardy-Littlewood inequality
\begin{equation}
\left\| \frac{1}{s} \int_0^s |f| \right\|_{L^2_t} \lesssim \|f\|_{L^2_t}
\end{equation}
it follows that
\begin{equation}
\|F\|_{L^2_t} \lesssim e^{C \Delta_0 \tau^{1/2}} \left\| \frac{1}{t(p) - t} \int_t^{t(p)} r |H| \right\|_{L^2_t} \lesssim e^{C \Delta_0 \tau^{1/2}} \|r H\|_{L^2_t}.
\end{equation}

Integrating (8.37) with respect to the angular variable $\omega \in S^2$ yields (8.34).

Next we multiply (8.36) by $r^2$ to obtain
\begin{equation}
\sup_{t(p) - \tau \leq t \leq t(p)} r^2 |F| \lesssim e^{C \Delta_0 \tau^{1/2}} \|r H\|_{L^2_t},
\end{equation}
which, by taking the $L^2_\omega$ norm, gives (8.35). \qed
In view of (8.35) and Lemma 8.2, we are able to prove the following estimates for $\hat{\chi}$.

**Lemma 8.3.** For $\hat{\chi}$ there hold the estimates
\begin{align}
\|r^{-1}\hat{\chi}\|_{L^2} + \|r^{1/2}\hat{\chi}\|_{L^2_x L^\infty_t} + \|\hat{\nabla}_L \hat{\chi}\|_{L^2_t} &\leq C, \\
\|\hat{\chi}\|_{L^2_x L^\infty_t} &\leq CE_0^{1/4}.
\end{align}

**Proof.** We will use the transport equation (8.36), i.e.
\[ \hat{\nabla}_L \hat{\chi} + \text{tr} \hat{\chi} = \alpha. \]
Recall that $r\hat{\chi} \to 0$ as $t \to t(p)$, see [13]. Recall also that $\|\alpha\|_{L^2_t} \leq C$, see Theorem 4.4. It then follows from Lemma 8.2 that
\[ \|r^{1/2}\hat{\chi}\|_{L^2_x L^\infty_t} + \|\hat{\chi}\|_{L^2_x L^\infty_t} \leq C. \]
Next we use (8.40) again to estimate $\|\hat{\nabla}_L \hat{\chi}\|_{L^2}$. With the help of the bootstrap assumption (BA2) and the comparability of $r$, $s$ and $t(p) - t$ given in Lemma 7.1, we have
\[ \|\text{tr} \chi \hat{\chi}\|_{L^2} \lesssim \|\text{tr} \chi \hat{\chi}\|_{L^2} + \|\alpha\|_{L^2} \leq C. \]
Thus, from (8.40) it follows
\[ \|\hat{\nabla}_L \hat{\chi}\|_{L^2} \lesssim \|\text{tr} \chi \hat{\chi}\|_{L^2} + \|\alpha\|_{L^2} \leq C. \]
We therefore complete the proof of (8.35).

By making use of (8.29) and (8.38) together with the bootstrap assumption (BA3) we obtain
\[ \|\hat{\chi}\|_{L^2_x L^\infty_t} \lesssim (\|\hat{\nabla}_L \hat{\chi}\|_{L^2} + \|r^{-1}\hat{\chi}\|_{L^2_x L^\infty_t}) \frac{1}{4} \|\hat{\chi}\|_{L^2_x L^\infty_t} \leq CE_0^{1/4} \]
which gives (8.39). \[ \square \]

Now we are ready to complete the proof of Proposition 12.

**Proof of Proposition 12.** We first prove (8.22). Let $|\bar{\pi}| := |\bar{\pi}|_{S_t}$. It is easy to check
\[ \hat{\nabla}_L (s^{-1}|\bar{\pi}|^2) + \text{tr} \chi s^{-1}|\bar{\pi}|^2 = s^{-1}(\text{tr} \chi - \frac{2}{s})|\bar{\pi}|^2 + s^{-2}|\bar{\pi}|^2 + 2s^{-1}\hat{\nabla}_L \bar{\pi} \cdot \bar{\pi}. \]
We integrate the above equation along the null cone $N^{-}(p, \tau)$. By Lemma 7.1, it is easy to see $\int_{S_t} s^{-1}|\bar{\pi}|^2 \to 0$ as $t \to t(p)$. Therefore, by integration by parts we obtain
\[ \int_{N^{-}(p, \tau)} (s^{-2}|\bar{\pi}|^2 + s^{-1}(\text{tr} \chi - \frac{2}{s})|\bar{\pi}|^2 + 2s^{-1}\hat{\nabla}_L \bar{\pi} \cdot \bar{\pi}) \text{nad} \mu_g \text{d}t = \int_{S_{t(p) - \tau}} s^{-1}|\bar{\pi}|^2. \]
By Lemma 7.1 and (7.9) in Proposition 11, we have
\[ \left| \int_{S_{t(p) - \tau}} s^{-1}|\bar{\pi}|^2 \right| \leq \|r^{-1/2}\bar{\pi}\|^2_{L^2(S_{t(p) - \tau})} \leq C. \]
By (BA2), Lemma 7.1 and (7.9),
\[ \left| \int_{N^{-}(p, \tau)} \text{nas}^{-1}(\text{tr} \chi - \frac{2}{s})|\bar{\pi}|^2 \text{d} \mu_g \text{d}t \right| \leq C E_0 \tau \leq C. \]
By (8.25) we have
\[
\left| \int_{\mathcal{N}^{-}(p, r)} s^{-1} \nabla L \bar{\pi} \cdot \bar{\pi} \nabla d\mu_s \, dt \right| \lesssim \|\nabla L \bar{\pi}\|_{L^2} s^{-1} \|\bar{\pi}\|_{L^2} \leq C s^{-1} \|\bar{\pi}\|_{L^2}.
\]
Therefore
\[
\|s^{-1} \bar{\pi}\|_{L^2}^2 \leq C + C \|s^{-1} \bar{\pi}\|_{L^2}
\]
which implies \( \|s^{-1} \bar{\pi}\|_{L^2} \leq C \). Consequently, in view of Lemma 7.1, (8.22) follows.

As a byproduct, we have from (BA2) and Lemma 7.1 that
\[
\|\text{tr} \bar{\pi}\|_{L^2} \lesssim \|s^{-1} \bar{\pi}\|_{L^2} + \|\text{tr} \tau - \frac{2}{s} \|_{L^\infty} \tau \|s^{-1} \bar{\pi}\|_{L^2} \leq C(1 + \varepsilon_0 \tau) \leq C.
\]

Next we will show (8.23), we will use the equation (8.26), i.e.
\[
\nabla \bar{\pi} = \nabla \bar{\pi} + \text{tr} \theta \cdot \bar{\pi} + \hat{\theta} \cdot \bar{\pi}.
\]

Using \( \theta_{AB} = -a \chi_{AB} + k_{AB} \), we have from (7.8) and (8.39) that
\[
\|\hat{\theta} \cdot \bar{\pi}\|_{L^2} \lesssim \|\bar{\pi}\|_{L^4} (\|k\|_{L^4} + \|\hat{\chi}\|_{L^4}) \leq C \left( E_0^{1/4} + 1 \right) \tau^{1/2} \leq C.
\]

Since \( \text{tr} \theta = -a \text{tr} \chi + \delta^{AB} k_{AB} \), we have from (7.8) and (8.42) that
\[
\|\text{tr} \theta \bar{\pi}\|_{L^2 L^2} \lesssim \|k\|_{L^4} \|\bar{\pi}\|_{L^4} + \|\text{tr} \chi \bar{\pi}\|_{L^2} \leq C.
\]

Consequently, in view of (8.25) and (8.43), (8.23) follows immediately.

In view of (8.27) and (8.20), (8.24) follows immediately from (8.25) and (7.8). \( \square \)

### 8.3. Estimates for Ricci coefficients.

**Lemma 8.4.** For the Ricci coefficient \( \zeta \) and the null lapse a there hold
\[
\|r^{1/2} \zeta\|_{L^2 L^\infty} + \|r^{-1} \zeta\|_{L^2} + \|\nabla L \zeta\|_{L^2} \leq C,
\]
\[
\|r^{1/2} \nabla \log a\|_{L^2 L^\infty} + \|r^{-1} \nabla \log a\|_{L^2} + \|\nabla L \nabla \log a\|_{L^2} \leq C.
\]

**Proof.** From the transport equation (8.6) we have
\[
\nabla L \zeta + \frac{1}{2} \text{tr} \chi \cdot \zeta = -\hat{\chi} \cdot \zeta + \chi \cdot \zeta - \beta.
\]

Since (BA3) implies \( \|\hat{\chi}\|_{L^\infty L^2} \leq E_0^{1/2} \) with \( E_0 \tau \leq 1 \), it follows from Lemma 8.2 and the relation \( \chi = \hat{\chi} + \frac{1}{2} \text{tr} \chi \gamma \) that
\[
\|r^{1/2} \zeta\|_{L^2 L^\infty} + \|r^{-1} \zeta\|_{L^2} \lesssim \|\beta\|_{L^2} + \|\hat{\chi} \cdot \zeta\|_{L^2} + \|\text{tr} \chi \cdot \zeta\|_{L^2}.
\]

From Theorem 4.4 we have \( \|\beta\|_{L^2} \leq C \). Recall that \( \zeta = \nabla \log n - \epsilon \) which is a combination of terms in \( \epsilon \). By (8.42) we have \( \|\text{tr} \chi \cdot \zeta\|_{L^2} \leq C \). Therefore
\[
\|r^{1/2} \zeta\|_{L^2 L^\infty} + \|r^{-1} \zeta\|_{L^2} \leq C (E_0 \tau + 1) + \|\hat{\chi} \cdot \zeta\|_{L^2}.
\]

In view of (7.8) in Proposition 11, (8.39) in Lemma 8.3 and \( E_0 \tau \leq 1 \), we have
\[
\|r^{1/2} \zeta\|_{L^2 L^\infty} + \|r^{-1} \zeta\|_{L^2} \leq C + r^{1/2} \|\hat{\chi}\|_{L^4 L^\infty} \|\zeta\|_{L^2 L^4} \leq C.
\]

Consequently, it follows from (8.40), (BA2) and (BA3) that \( \|\nabla L \zeta\|_{L^2} \leq C \). We thus obtain (8.44).

In order to show (8.45), we use the relation \( \zeta = \nabla \log a + \epsilon \). By Proposition 12
\[
\|r^{1/2} \epsilon\|_{L^2 L^\infty} + \|\epsilon\|_{L^2 L^2} + \|\nabla L \epsilon\|_{L^2} \leq C.
\]
Thus, the estimates for $\nabla \log a$ follows. \hfill \Box

**Lemma 8.5.** For the $\mu$ defined by (8.11) there holds $\|\mu\|_{L^2} \leq C$ on $\mathcal{N}^{-}(p, \tau)$.

**Proof.** Recall that by (8.7), $\mu = 2 \text{div} \xi - \hat{\chi} \cdot \hat{\eta} + 2|\xi|^2 + 2 \rho$. We have from Theorem 4.4, Proposition 11 and Theorem 4.5 that

$$
\|\mu\|_{L^2} \lesssim \|\nabla \xi\|_{L^2} + \|\xi\|_{L^4}^2 + \|\rho\|_{L^2} + \|\hat{\chi} \cdot \hat{\eta}\|_{L^2} \lesssim C + \|\hat{\chi} \cdot \hat{\eta}\|_{L^2}.
$$

Recall also the relation $\hat{\chi}' = -\chi' - 2\hat{\eta}$, we have from (8.39) and Proposition 11 that

$$
\|\mu\|_{L^2} \lesssim C + \|\hat{\chi}\|_{L^4} (\|\hat{\chi}\|_{L^4} + \|k\|_{L^4}) \leq C.
$$

The proof is thus complete. \hfill \Box

In the following we summarize the estimates obtained so far in this section.

**Proposition 13.** There exists universal constants $\delta_0 > 0$ and $C_* > 0$ such that, under the bootstrap assumptions (BA1)-(BA3) with $\mathcal{E}_0 \tau \leq 1$, if $\tau < \min\{i_*, \delta_0\}$ then there hold

$$
\|r^{-\frac{1}{2}} \xi\|_{L^2(S_t, \omega)} \leq C, \tag{8.47}
$$

$$
\|\xi\|_{L^4(S_t, \omega)} \leq C, \tag{8.48}
$$

$$
\mathcal{N}_1[\xi](p, \tau) \leq C, \tag{8.49}
$$

$$
\|n^{-1} \nabla^2 n, n^{-2} \nabla n\|_{L^2} \leq C, \tag{8.50}
$$

$$
\|r^{\frac{1}{2}}(\hat{\chi}, \hat{\pi}, \xi, \nabla \log a, \hat{\theta})\|_{L^2 L^\infty} \leq C, \tag{8.51}
$$

$$
\|(\hat{\chi}, \hat{\pi}, \xi, \nabla \log a, \hat{\theta})\|_{L^2 L^2} \leq C, \tag{8.52}
$$

$$
\|\nabla_L (\hat{\chi}, \hat{\pi}, \xi, \nabla \log a, \hat{\theta})\|_{L^2} \leq C, \tag{8.53}
$$

where $\hat{\pi} = (n^{-1} \partial_t \log n, \hat{\pi})$.

The above estimates provide the intermediate steps toward the proof of Theorem 4.6. The complete proof however requires more estimates on $\hat{\chi}, \xi$ and $\xi$ as follows. Since the arguments are rather lengthy, we will report them in [14].

**Proposition 14.** There exists universal constants $\delta_0 > 0$ and $C_* > 0$ such that, under the bootstrap assumptions (BA1)-(BA4) with $\mathcal{E}_0 \tau \leq 1$, if $\tau < \min\{i_*, \delta_0\}$ then there hold

$$
\left\| \text{tr} \chi - \frac{2}{s} \right\|_{L^\infty} \leq C_*, \tag{8.54}
$$

$$
\|\hat{\chi}\|_{L^\infty L^2} + \|\xi\|_{L^\infty L^4} \leq C_*, \tag{8.55}
$$

$$
\|\nu\|_{L^\infty L^2} + \|\xi\|_{L^\infty L^2} \leq C_*, \tag{8.56}
$$

$$
\mathcal{N}_1[\hat{\chi}, \xi, \nabla \log a, \hat{\theta}](p, \tau) \leq C_*, \tag{8.57}
$$

$$
\|r^{\frac{1}{2}}(\nabla \text{tr} \chi, \mu)\|_{L^2 L^\infty} + \|(\nabla \text{tr} \chi, \mu)\|_{L^2} \leq C_*, \tag{8.58}
$$

on the null cone $\mathcal{N}^{-}(p, \tau)$ for all $p \in \mathcal{M}_I$.

The estimates in Proposition 13 and Proposition 14 gives Theorem 4.6. Thus, we may use a bootstrap argument, as explained in Section 4, to conclude that all the estimates in the above two propositions hold on the null cones $\mathcal{N}^{-}(p, \tau)$ for all $p \in \mathcal{M}_I$ with $\tau = \min\{i_*, \delta_0\}$ for some universal constant $\delta_0 > 0$. 

We conclude this section with an application to estimate $\| \pi \|_{L^2_t L^2_x(\text{Int}(S_{t,u})))}$, where, for any $\Sigma$ tangent tensor $F$,

$$
\| F \|_{L^2_t L^2_x(\text{Int}(S_{t,u})))}^2 = \int_{u_m}^u \int_{S_{t,u'}} r'^{-2} |F|_g^2 \, d\mu_g \, du'
$$

with $r' = r(t, u')$.

**Proposition 15.** For $\pi = (n^{-1} \partial_t \log n, \pi)$, there holds

$$(8.59) \quad \| \pi \|_{L^2_t L^2_x(\text{Int}(S_{t,u})))} \leq C.$$ 

**Proof.** From (8.4), (8.10), (8.12) and (8.8), we can derive

$$(8.60) \quad \nabla_N \text{tr} \chi' + \frac{1}{2} (\text{tr} \chi')^2 = -\frac{1}{2} \delta \text{tr} \chi' + 2 \lambda \text{tr} \chi' - \hat{\chi}'(\hat{\chi} + \hat{n}) - (\text{div} \, \bar{\zeta} + \| \psi \|^2 + \rho),$$

which, multiplied by $|\pi| := |\pi|_g$, implies

$$
\nabla_N (\text{tr} \chi |\pi|_g^2) + \text{tr} \theta (\text{tr} \chi |\pi|_g^2) = \frac{1}{2} \text{tr} \chi |\pi|_g^2
$$

$$
= \left\{ -\frac{3}{2} \delta \text{tr} \chi' - \hat{\chi}'(\hat{\chi} + \hat{n}) - (\text{div} \, \bar{\zeta} + \| \psi \|^2 + \rho) \right\} |\pi|_g^2 + 2 \text{tr} \chi \nabla_N \pi \cdot \bar{\pi}
$$

In view of Lemma 7.2, integrating the above equation over $\text{Int}(S_{t,u})$ gives

$$
\frac{1}{2} \int_{u_m}^u \int_{S_{t,u'}} (\text{tr} \chi')^2 |\pi|_g^2 \, d\mu_g \, du'
$$

$$
= -\int_{S_{t,u}} \text{tr} \chi' |\pi|_g^2 + \int_{u_m}^u \int_{S_{t,u'}} (-2 \nabla_N \pi \cdot \text{tr} \chi' \pi + \rho |\pi|_g^2) \, d\mu_g \, du'
$$

$$
+ \int_{u_m}^u \int_{S_{t,u'}} \left( \frac{3}{2} \delta \text{tr} \chi' + \| \psi \|^2 + \hat{\chi}'(\hat{\chi} + \hat{n}) \right) |\pi|_g^2 \, d\mu_g \, du'
$$

$$
(8.61) \quad + \int_{u_m}^u \int_{S_{t,u'}} -\zeta \cdot \nabla (|\pi|^2 a) \, d\mu_g \, du'
$$

By (BA2), Lemma 7.1 and (5.19),

$$
\left| \int_{S_{t,u}} \text{tr} \chi' |\pi|_g^2 \, d\mu_g \right| \lesssim \| r^{-1/2} \pi \|_{L^2(\text{Int}(S_{t,u}))} \leq C.
$$

By Lemma 2.2, Proposition 8 and (5.11),

$$
\left| \int_{u_m}^u \int_{S_{t,u'}} \nabla_N \pi \cdot \text{tr} \chi' \pi \, d\mu_g \, du' \right| \lesssim \| \nabla_N \pi \|_{L^2(\Sigma_t)} \| \text{tr} \chi' \pi \|_{L^2(\Sigma_t)} \leq C \| \text{tr} \chi' \pi \|_{L^2(\Sigma_t)}
$$

and

$$
\left| \int_{u_m}^u' \int_{S_{t,u}} \frac{3}{2} \delta \text{tr} \chi' |\pi|_g^2 \, d\mu_g \, du' \right| \lesssim \| k \|_{L^6(\Sigma_t)} \| \pi \|_{L^6(\Sigma_t)}^2 + \| \pi \|_{L^6(\Sigma_t)}^3 \lesssim (\| \nabla k \|_{L^2(\Sigma_t)} + \| \pi \|_{H^1(\Sigma_t)}) \| \pi \|_{H^1(\Sigma_t)}^2 \leq C.
$$
Therefore, we obtain
\[ \left| \int_{u_m}^u \int_{S_{t,u'}} \rho |\pi|^2 a d\mu_\tau d\mu' \right| \lesssim \|\rho\|_{L^2(\Sigma_t)} \|\pi\|_{L^4(\text{Int}S_{t,u'})}^2 \leq C(u-u_m)^{1/2}. \]
By (8.19) we have
\[ \left| \int_{u_m}^u \int_{S_{t,u'}} \xi \nabla (a|\pi|^2) d\mu_\tau d\mu' \right| = \int_{u_m}^u \int_{S_{t,u'}} (\nabla \log a|\pi|^2 \xi + \nabla |\pi|^2 \xi) d\mu_\tau d\mu' \lesssim \|\nabla \pi\|_{L^2(\text{Int}(S_{t,u}))} \sup_{u_m \leq u' \leq u} \left( \|\pi\|_{L^4(S_{t,u'})} \|\xi\|_{L^4(S_{t,u'})} \right) (u-u_m)^{1/2} + \int_{u_m}^u \int_{S_{t,u'}} (|\xi|^2 |\pi|^2 + |\xi| |\pi|^3) d\mu_\tau d\mu'. \]
In view of Lemma 2.2, Proposition 4 and (8.10) we derive
\[ \|\nabla \pi\|_{L^2(\text{Int}(S_{t,u}))} \leq \|\nabla \pi\|_{L^2(\Sigma_t)} \leq C, \]
while in view of (8.57), (8.28) and (7.8) we have
\[ \sup_{u_m \leq u' \leq u} \|\xi\|_{L^4(S_{t,u'})} \leq C, \quad \sup_{u_m \leq u' \leq u} \|\pi\|_{L^4(S_{t,u'})} \leq C. \]
Consequently,
\[ \int_{u_m}^u \int_{S_{t,u'}} (|\xi|^2 |\pi|^2 + |\xi| |\pi|^3) d\mu_\tau d\mu' \lesssim \sup_{u_m \leq u' \leq u} \left( \|\xi\|_{L^4(S_{t,u'})}^2 \|\pi\|_{L^4(S_{t,u'})}^2 \right) (u-u_m) + \sup_{u_m \leq u' \leq u} \left( \|\xi\|_{L^4(S_{t,u'})} \|\pi\|_{L^4(S_{t,u'})}^3 \right) (u-u_m) \leq C(u-u_m). \]
Therefore, we obtain
\[ \left| \int_{u_m}^u \int_{S_{t,u'}} \xi \nabla (a|\pi|^2) d\mu_\tau d\mu' \right| \leq C(1 + (u-u_m)^{1/2})(u-u_m)^{1/2}. \]
In view of (8.57), (8.28) and (7.8), by a similar argument we obtain
\[ \left| \int_{u_m}^u \int_{S_{t,u'}} (|\xi|^2 + \tilde{\chi}(\tilde{\chi}' + \tilde{\eta}))) |\pi|^2 d\mu_\tau d\mu' \right| \lesssim \int_{u_m}^u \int_{S_{t,u'}} (|\tilde{\pi}|^2 (|\tilde{\chi}|^2 + |\xi|^2) + |\tilde{\chi}| |\tilde{\pi}|^3) d\mu_\tau d\mu' \leq C(u-u_m) \]
Combining all the above estimates with (8.61) and noting that \( u-u_m \lesssim \tau \lesssim 1 \), it yields
\[ \|\text{tr} \pi\pi\|_{L^2(\text{Int}(S_{t,u}))} \leq C + C\|\text{tr} \pi\pi\|_{L^2(\text{Int}(S_{t,u}))} \]
which implies \( \|\text{tr} \pi\pi\|_{L^2(\text{Int}(S_{t,u}))} \leq C \). This together with (BA2) implies the desired inequality. \( \square \)
9. Proof of Theorem 4.7

In this section we will complete the proof of Theorem 4.7. For any \( p \in M_t \), let \( \Phi(t) \) be the integral curve of \( T \) through \( p \) with \( \Phi(t(p)) = p \). For each \( p_t := \Phi(t) \), we will represent \( k(p_t) \) in terms of a Kirchoff-Sobolev formula over a past null cone with vertex \( p_t \). We then use the estimates established in the previous section to obtain \( \int_{t(p) - \tau}^{t(p)} |k(\Phi(t))|^2 n dt \leq C \) for some universal constant \( C \).

9.1. Derivation of Kirchoff Parametrix. We first revisit the formulation of Kirchoff Parametrix in [10]. We define \( A \) to be a \( \Sigma_t \) tangent 2-tensor verifying

\[
(D_L A)_{ij} + \frac{1}{2} \text{tr} \chi A_{ij} = 0 \quad \text{on } N^{-}(p, \tau), \quad \lim_{t \to t(p)^{-}} (t(p) - t) A_{ij} = J_{ij},
\]

where \( J \in T_p \Sigma_t(p) \) and \( |J|_g = 1 \). This \( A \) is similar to the one defined in [12] but with the modification that \( A \) is \( \Sigma_t \) tangent. Since we have obtained in Propositions 13 and 14 the estimates on

\[
\| \text{tr} \chi - \frac{2}{s} \|_{L^\infty}, \quad \| \nabla \text{tr} \chi \|_{L^2}, \quad \| r^2 \nabla \text{tr} \chi \|_{L^2 (\Sigma)}, \quad \| r^{-1} (\zeta + \zeta') \|_{L^2}, \quad \| \chi, \nu, \zeta \|_{L^2 (\Sigma)}, \quad \mathcal{R}(p, \tau)
\]
on the null cone \( N^{-}(p, \tau) \), we may adapt the proof in [12] to obtain the following estimates on \( A \).

Proposition 16. For the tensor \( A \) defined by (9.1) there hold

\[
\| \nabla A \|_{L^2(N^-(p, \tau))} + \| r^2 \nabla A \|_{L^2 (\Sigma N^-(p, \tau))} + \| r A \|_{L^\infty(N^-(p, \tau))} \leq C,
\]

where \( C \) is a universal constant.

Now we revisit the Kirchoff-Sobolev formula for any \( \Sigma_t \) tangent 2-tensor \( \Psi_I \), \( I = \{\mu, \nu\} \), see [10 15]. According to the definition of \( \Box \Psi_I \), we have under the null frame \( (e_A)_{A=1,2}, e_3 = L, e_4 = L \) that

\[
\Box \Psi_I = -\frac{1}{2} D_{43} \Psi_I - \frac{1}{2} D_{34} \Psi_I + \delta^{AB} D_{AB} \Psi_I.
\]

By (8.1),

\[
D_{43} \Psi_I = D_4 (D_3 \Psi_I) - 2 \zeta A D_A \Psi_I.
\]

It is easy to see

\[
D_{34} \Psi_I - D_{43} \Psi_I = R_{\mu}^{\alpha} \Psi_I + R_{\nu}^{\alpha} \Psi_I.
\]

By (8.2), we obtain

\[
\delta^{AB} D_{AB} \Psi_I = \delta^{AB} \nabla_A \nabla_B \Psi_I - \frac{1}{2} \text{tr} \chi D_4 \Psi_I - \frac{1}{2} \text{tr} \chi D_3 \Psi_I.
\]

Therefore

\[
\Box \Psi_I = -D_4 (D_3 \Psi_I) + 2 \zeta A D_A \Psi_I - \frac{1}{2} \text{tr} \chi D_4 \Psi_I - \frac{1}{2} \text{tr} \chi D_3 \Psi_I + \delta^{AB} \nabla_A \nabla_B \Psi_I - \frac{1}{2} R_{\mu}^{\alpha} \Psi_I - \frac{1}{2} R_{\nu}^{\alpha} \Psi_I.
\]
We multiply the above equation by $A_I$ and integrate over $N^-(p, \tau)$ to obtain

$$\int_{N^-(p, \tau)} \Box \Psi_I A^I = \Xi_1 + \Xi_2 + \int_{N^-(p, \tau)} \left( 2 \varepsilon^{AB} D_A \Psi_I \cdot A^I + \delta^{AB} \nabla_A \nabla_B \Psi_I \cdot A^I \right)$$

(9.4)

$$- \frac{1}{2} \int_{N^-(p, \tau)} (R_{\mu}{}_{\alpha}{}^\alpha \Psi_{\mu}{}^\alpha + R_{\nu}{}_{\alpha}{}^\alpha \Psi_{\mu}{}^\alpha) A^{\mu \nu}.$$ 

where

$$\Xi_1 = \int_{N^-(p, \tau)} \left( -D_4 (D_3 \Psi)_I \cdot A^I - \frac{1}{2} \text{tr} D_3 \Psi_I \cdot A^I + \frac{1}{2} \text{tr} \chi D_3 \Psi_I \cdot A^I \right),$$

$$\Xi_2 = - \frac{1}{2} \int_{N^-(p, \tau)} \text{tr} \chi D_4 \Psi_I \cdot A^I.$$ 

For $\Xi_1$, integrating by parts gives

$$\Xi_1 = \int_{N^-(p, \tau)} \left( -D_4 (D_3 \Psi)_I \cdot A^I - \text{tr} \chi D_3 \Psi_I \cdot A^I + \frac{1}{2} \text{tr} \chi D_3 \Psi_I \cdot A^I \right)$$

$$= - \int_{S_t(p, \tau)} D_3 \Psi_I \cdot A^I + \lim_{t \to t(p)} \int_{S_t} D_3 \Psi_I \cdot A^I$$

$$+ \int_{N^-(p, \tau)} \left( D_4 A^I + \frac{1}{2} \text{tr} \chi A^I \right) \cdot D_3 \Psi_I$$

Since $\lim_{t \to t(p)} (t(p) - t)^2 A = 0$, we have in view of (9.1) that

$$\Xi_1 = - \int_{S_t(p, \tau)} D_3 \Psi_I \cdot A^I + \int_{N^-(p, \tau)} \Omega_1(\Psi),$$

where

$$\Omega_1(\Psi) = D_4 A_{\alpha t} \cdot D_3 \Psi_{\alpha t} + D_4 A_{\alpha 0} \cdot D_3 \Psi_{\alpha 0}.$$ 

For the term $\Xi_2$, in view of (9.1) and the fact that $\Psi$ is $\Sigma_t$ tangent, we first have

$$- \frac{1}{2} \text{tr} \chi D_4 \Psi_I \cdot A^I = \frac{1}{2} \left( D_4 (\Psi_I \cdot A^I \text{tr} \chi) - D_4 A^I \cdot \text{tr} \chi \cdot \Psi_I - D_4 \text{tr} \chi \cdot \Psi_I \cdot A^I \right)$$

$$= - \frac{1}{2} \left( D_4 (\Psi_I \cdot A^I \text{tr} \chi) + \frac{1}{2} \text{tr} \chi A^I \cdot \Psi_I - D_4 \text{tr} \chi \cdot \Psi_I \cdot A^I \right),$$

thus integration by parts yields

$$\Xi_2 = \int_{N^-(p, \tau)} \frac{1}{2} \mu A^I \cdot \Psi_I - \frac{1}{2} \left( \int_{S_t(p, \tau)} \Psi_I \cdot A^I \text{tr} \chi - \lim_{t \to t(p)} \int_{S_t} \Psi_I \cdot A^I \text{tr} \chi \right),$$

where $\mu$ is defined in (8.11).

In view of $\text{tr} \chi = - \text{tr} \chi' - 2 \delta^{AB} k_{AB}$, we have

$$\lim_{t \to t(p)} \frac{1}{2} \int_{S_t} \Psi_I \cdot A^I \text{tr} \chi = -4 \pi n(p) (\Psi, J).$$

Hence

$$\Xi_2 = \int_{N^-(p, \tau)} \frac{1}{2} \mu A^I \cdot \Psi_I - \frac{1}{2} \int_{S_t(p, \tau)} \Psi_I \cdot A^I \text{tr} \chi - 4 \pi n(p) (\Psi, J).$$
Therefore we derive

\[
4\pi n(p) \langle \Psi, J \rangle = \int_{\mathcal{N}^-(p, \tau)} \left( -\Box \Psi_I \cdot A^I + \frac{1}{2} \mu \Psi_I \cdot A^I + \Omega_1(\Psi) \right) - \int_{S_t(p) - \tau} \left( D_3 \Psi_I \cdot A^I + \frac{1}{2} \text{tr} \chi \Psi_I \cdot A^I \right) + \int_{\mathcal{N}^-(p, \tau)} \left( 2 \zeta B \Psi_I \cdot A^I - \nabla^B \Psi_I \cdot \nabla^B A^I \right) - \frac{1}{2} \int_{\mathcal{N}^-(p, \tau)} \left( R_{a_{34}} \Psi_{a_{10}} + R_{a_{34}} \Psi_{a_{10}} A^{ij} \right).
\]

(9.5)

We apply (9.5) to the tensor field \( \Psi = k \) and obtain

**Theorem 9.1.** Let \( p \in \mathcal{M}_1 \), let \( \Phi(t) \) be the integral curve of \( T \) through \( p \) with \( \Phi(t(p)) = p \), and let \( p_t = \Phi(t) \). Let \( A \) be a \( \Sigma \) tangent 2-tensor on \( \mathcal{J}^- (p, \tau) \) verifying (9.1) on each null cone \( C_u := \mathcal{N}^- (p_t, t - t(p) + \tau) \), where \( u = u(t) = \int_{t_0}^t n_4 \mu \omega dt \) for \( t_m := t(p) - \tau \leq t \leq t(p) \). Then there holds

\[
4\pi n(p_t) \langle k(p_t), J \rangle = I(p_t) + J(p_t) + K(p_t) + L(p_t) + \mathfrak{C}(p_t) + \int_{C_u} \Omega_1(k),
\]

where

\[
I(p_t) = -\int_{C_u} A \cdot k,
\]

\[
J(p_t) = -\frac{1}{2} \int_{C_u} A \cdot R(\cdot, \cdot, L, L) \cdot k,
\]

\[
K(p_t) = \int_{C_u} \left( -\nabla^B \cdot \nabla^B k + 2 \zeta B \cdot \nabla^B k \cdot A \right),
\]

\[
L(p_t) = \frac{1}{2} \int_{C_u} \mu A \cdot k,
\]

\[
\mathfrak{C}(p_t) = -\int_{S_{t_m} \setminus C_u} \left( D_3 k \cdot A + \frac{1}{2} \text{tr} \chi k \cdot A \right).
\]

9.2. **Main estimates.** In the following we will use the representation formula given in Theorem 9.1 to show that

\[
\int_{t(p) - \tau}^{t(p)} |k(p_t)|^2 n dt \leq C
\]

for some universal constant \( C \). We proceed as follows.

- **Estimate on \( I(p_t) \):** We will use the expression of \( \Box k \) given in Proposition 8, which symbolically can be written as

\[
\Box k = -n^{-3} \mu \nabla^2 n + n^{-2} \nabla^2 n + \pi \cdot \pi + k \cdot \nabla^2 n + k \cdot Ric + \pi \cdot \nabla k - n^{-1} k.
\]
It then follows from Proposition [16] that

\[ |I(p_t)| \lesssim \int_{C_u} r^{-1} \left( |\hat{n} \nabla^2 u| + |\nabla^2 \hat{n}| + |\pi|^3 + |k||\nabla^2 n| + |k||Ric| + |\pi||\nabla k| + |k| \right) \]

\[ \lesssim \|\nabla^2 n\|_{L^2(C_u)} \|r^{-1}\hat{n}\|_{L^2(C_u)} + \|r^{-1}\nabla^2 \hat{n}\|_{L^1(C_u)} + \int_{C_u} r^{-1}|\pi|^3 \]

\[ + \|r^{-1}k\|_{L^2(C_u)} \|\nabla^2 n\|_{L^2(C_u)} + \|Ric\|_{L^2(C_u)} \|r^{-1}k\|_{L^2(C_u)} \]

\[ + \|r^{-1}\pi\|_{L^2(C_u)} \|\nabla k\|_{L^2(C_u)} + \|r^{-1}k\|_{L^1(C_u)}. \]

Therefore, with the help of Proposition [11] and Proposition [12] we have

\[ |I(p_t)| \lesssim \|r^{-1}\hat{n}\|_{L^2(C_u)} + \|r^{-1}\nabla^2 \hat{n}\|_{L^1(C_u)} + \int_{C_u} r^{-1}|\pi|^3 \]

\[ + \|Ric\|_{L^2(C_u)} + \|\nabla k\|_{L^2(C_u)} + C. \]

Now we consider \( \int_{t_m}^{t(p)} |I(p_t)|^2 dt \). Using \( \frac{du}{dt} = n \), we have from Proposition [15] that

\[ \int_{t_m}^{t(p)} \|r^{-1}\hat{n}\|_{L^2(C_u)}^2 dt = \int_{t_m}^{u(t(p))} \|r^{-1}\hat{n}\|_{L^2(C_u)}^2 du \]

\[ = \int_{t_m}^{u(t(p))} \int_{t_m}^{t_{Mf}(u)} \int_{S_{t',u}} r^{-2}|\hat{n}|^2 n d\mu dtdt' \]

\[ = \int_{t_m}^{t(p)} \int_{t_m}^{u(t(p))} \int_{S_{t',u}} r^{-2}|\hat{n}|^2 n d\mu dtdt' \]

\[ \lesssim \int_{t_m}^{t(p)} \|r^{-1}\hat{n}\|_{L^2(\text{Int}(S_{t',u(t(p))}))}^2 dt' \]

\[ \leq C\tau. \]

By similar argument, we have from Lemma [2.2] that

\[ \int_{t_m}^{t(p)} \left( \|Ric\|_{L^2(C_u)}^2 + \|\nabla k\|_{L^2(C_u)}^2 \right) n dt \leq C\tau. \]

Therefore

\[ \int_{t_m}^{t(p)} |I(p_t)|^2 dt \leq C\tau + \int_{t_m}^{t(p)} \|r^{-1}\nabla^2 \hat{n}\|_{L^1(C_u)}^2 n dt + \int_{t_m}^{t(p)} \left( \int_{C_u} r^{-1}|\pi|^3 \right)^2 n dt. \]

By using the Minkowski inequality and Proposition [7] we have

\[ \left( \int_{t_m}^{t(p)} \|r^{-1}\nabla^2 \hat{n}\|_{L^1(C_u)}^2 n dt \right)^{1/2} \]

\[ = \left( \int_{t_m}^{u(t(p))} \left( \int_{t_m}^{t_{Mf}(u)} r^{-1}\|\nabla \nabla^2 \hat{n}\|_{L^2(S_{t',u})}^2 dtdt'\right)^{1/2} du \right) \]

\[ \leq \int_{t_m}^{t(p)} \left( \int_{u(t')}^{u(t(p))} r^{-2}\|\nabla \nabla^2 \hat{n}\|_{L^2(S_{t',u})}^2 dtdt' \right)^{1/2} \]

\[ \lesssim \int_{t_m}^{t(p)} \|\nabla^2 \hat{n}\|_{L^2(\text{Int}(S_{t',u(t(p))}))}^2 dt' \leq C. \]
Finally, we have from Proposition 11 and (8.22) that
\[
\int_{C_n} r^{-1} |\pi|^3 \lesssim \int_{C_m} r^{-1} \pi_{L^2(S_{t',w})}^2 dt' \leq C(t_M(u) - t_m)^{1/2}.
\]
Thus, by Lemma 7.1 we obtain
\[
\int_{C_n} t_m^{(p)} \left( \int_{C_n} r^{-1} |\pi|^3 \right)^2 dt \leq C r^2.
\]
Combining the above estimates we therefore obtain
\[
\int_{C_n} t_m^{(p)} |I(p_t)|^2 dt \leq C + Cr^2 \lesssim C.
\]

- **Estimate on** \(J(p_t)\): It follows from Proposition 16, Theorem 4.4 and Proposition 12 that
\[
|J(p_t)| \lesssim \|rA\|_{L^\infty(C_n)} \|r^{-1} \pi\|_{L^2(C_n)} R(p_t, \tau + t - t(p)) \leq C.
\]
Thus
\[
\int_{t_m}^{t(p)} |J(p_t)|^2 dt \leq C(t(p) - t_m) \leq C \tau \leq C.
\]

- **Estimate on** \(K(p_t)\): It follows from the Hölder inequality that
\[
|K(p_t)| \lesssim \|\nabla A\|_{L^2(C_n)} \|\nabla \pi\|_{L^2(C_n)} + \|rA\|_{L^\infty(C_n)} \|r^{-1} \pi\|_{L^2(C_n)} \|\nabla k\|_{L^2(C_n)}.
\]
Thus, we obtain from Proposition 16, Theorem 4.4 and Proposition 12 that \(|K(p_t)| \leq C\) which gives
\[
\int_{t_m}^{t(p)} |K(p_t)|^2 dt \leq C(t(p) - t_m) \leq C \tau \leq C.
\]

- **Estimate on** \(L(p_t)\): It follows from Proposition 16 and Proposition 12 that
\[
|L(p_t)| \leq \|rA\|_{L^\infty(C_n)} \|r^{-1} \bar{\pi}\|_{L^2(C_n)} \|A\|_{L^2(C_n)} \lesssim \|A\|_{L^2(C_n)}.
\]
From Lemma 8.3 we then obtain \(|L(p_t)| \leq C\). Therefore
\[
\int_{t_m}^{t(p)} |L(p_t)|^2 dt \leq C(t(p) - t_m) \leq C \tau \leq C.
\]

- **Estimate on** \(E(p_t)\): We first have from Proposition 16 that
\[
|E(p_t)| \lesssim r^{-1} \|D^\delta k\|_{L^1(S_{t',w})} + r^{-1} \|\text{tr} k\|_{L^1(S_{t',w})}.
\]
Using the definition of \(r\) we then obtain
\[
|E(p_t)| \lesssim \|D^\delta k\|_{L^1(S_{t',w})} + r^{-1} \|\text{tr} k\|_{L^1(S_{t',w})}.
\]
Recall
\[
\text{tr} k' = -\text{tr} k - 2\delta^{AB} k_{AB}.
\]
Since \((BA1)\) implies \(1/2 \leq a \leq 3/2\). Thus, with the help of \((BA2)\), it yields
\[
\|\text{tr} k\|_{L^1(S_{t',w})} \lesssim \|\text{tr} - \frac{2}{s} \|_{L^\infty(C_n)} \|k\|_{L^1(S_{t',w})} + r^{-1} \|k\|_{L^1(S_{t',w})} + \|k\|_{L^2(S_{t',w})}^2
\]
\[
\lesssim r^{-1} \|k\|_{L^1(S_{t',w})} + \|k\|_{L^2(S_{t',w})}^2
\]
\[
\lesssim \|k\|_{L^2(S_{t',w})} + r \|k\|_{L^2(S_{t',w})}^2.
\]
Consequently
\[ |\mathcal{E}(p_t)| \lesssim \|D_3 k\|_{L^2(S_{t_m,u})} + r^{-1} \|k\|_{L^2(S_{t_m,u})} + \|k\|_{L^4(S_{t_m,u})}^2. \]

Therefore, using \( \frac{du}{dt} = n \), we have
\[ \int_{t_m}^{t(p)} |\mathcal{E}(p_t)|^2 dt \lesssim \int_{u(t_m)}^{u(t(p))} |\mathcal{E}(p_t)|^2 du \]
\[ \lesssim ||D_3 k||_{L^2(\Sigma_{t_m})} + ||r^{-1}k||_{L^2(Inter(S_{t_m,u}))} + ||k||_{L^4(\Sigma_{t_m})}^2. \]

It follows from Lemma 2.2 and Proposition 3 that

\[ \int_{t_m}^{t(p)} |\mathcal{E}(p_t)|^2 dt \lesssim ||D_3 k||_{L^2(\Sigma_{t_m})} + C. \]

Recall that \( L = -a(T - N) \). So \( D_3 k = -a(D_0 k - \nabla_N k) \). Recall also that \( D_0 k = -n^{-1} \nabla^2 n + R_{ik} + kT_{ik} \). Thus
\[ ||D_3 k||_{L^2(\Sigma_{t_m})} \lesssim ||\nabla^2 n||_{L^2(\Sigma_{t_m})} + ||R_{ik}||_{L^2(\Sigma_{t_m})} + ||k||_{L^4(\Sigma_{t_m})}^2. \]

It follows from Lemma 2.2 and Proposition 3 that \( ||D_3 k||_{L^2(\Sigma_{t_m})} \leq C \). Therefore
\[ \int_{t_m}^{t(p)} |\mathcal{E}(p_t)|^2 dt \lesssim ||D_3 k||_{L^2(\Sigma_{t_m})}^2 n dt \leq C. \]

- **Estimate on \( \int_{C_u} \Omega_1(k) \)**: By straightforward calculation we have \( \Omega_1(k) = A \cdot \bar{\pi} \cdot \bar{\pi} \cdot \bar{\pi} \). It follows from Proposition 16 that
\[ |\Omega_1(k)| \lesssim \int_{C_u} r^{-1} |\bar{\pi}|^3. \]

Therefore, one can use the similar argument in the estimate of \( I(p_t) \) to get
\[ \int_{t_m}^{t(p)} |\Omega_1(k)|^2 n dt \lesssim \int_{t_m}^{t(p)} \left( \int_{C_u} r^{-1} |\bar{\pi}|^3 \right)^2 n dt \leq C \tau^2 \leq C. \]

10. **Proof of main theorem I**

In this section, based on Theorem 1.2, we will follow the idea in [12] to give the proof of Theorem 1.2. According to the local existence theorem given in [12, Proposition 6.1], see also [5, Theorem 10.2.1], it suffices to show that the quantity
\[ \mathcal{R}_* := ||Ric||_{H^2(\Sigma_t)} + ||k||_{H^2(\Sigma_t)} \]
on each slice \( \Sigma_t \) with \( t_0 \leq t < t_* \) is uniformly bounded.

Since \( (M, g) \) is a vacuum space-time, by virtue of the Bianchi identity \( R \) verifies a wave equation of the form
\[ \Box R = R \cdot R, \]

Based on higher energy estimates it is standard to show that
\[ ||D\mathcal{R}(t)||_{L^2} \lesssim ||D\mathcal{R}(t_1)||_{L^2} + \int_{t_1}^{t} ||\mathcal{R}(t')||_{L^2} dt' \]
(10.3) and
\[ ||D^2\mathcal{R}(t)||_{L^2} \lesssim ||D^2\mathcal{R}(t_1)||_{L^2} + \int_{t_1}^{t} ||D\mathcal{R}(t')||_{L^2} ||\mathcal{R}(t')||_{L^2} dt' \]
(10.4)
for all $t_0 \leq t_1 \leq t \leq t_*$. The derivation has been given in [12] under the assumption [1.9], the argument however depends only on the condition (A1).

Thus, the derivation of the $L^\infty$ bound of $R$ is a crucial step. As in [10] one can represent $R(p)$, for each $p \in M_*$, by a Kirchoff-Sobolev formula over the null cone $N^-(p, \tau)$, where $\tau > 0$ is a universal constant such that $i_+(p, t) \geq \tau$ whose existence is guaranteed by Theorem 1.2. One can then follow the delicate argument in [12] to derive that

$$
(10.5) \quad \|R(t)\|_{L^\infty} \lesssim \tau^{-1} \sup_{t' \in \{t - 2\tau, t - \tau/2\}} (\|R(t')\|_{L^2} + \|D^2 R(t')\|_{L^2}).
$$

The derivation of (10.5) requires the estimates on

$$
R(p, \tau), \quad \|\text{tr} \chi - \frac{2}{8}\|_{L^\infty(N^-(p, \tau))}, \quad \|\hat{\chi}, \nu, \zeta, \xi\|_{L^\infty(N^-(p, \tau))},
$$

$$
\|\mu, \nabla \text{tr} \chi\|_{L^2(N^-(p, \tau))}, \quad \|\tau^{1/2} \nabla \text{tr} \chi\|_{L^2(N^-(p, \tau))}, \quad \|\zeta^{-1} (\zeta + \bar{\zeta})\|_{L^2(N^-(p, \tau))}
$$

which are provided by Proposition [13] and Proposition [14] under the condition (A1). Combining the estimates (10.3)–(10.5) gives

$$
\|R(t)\|_{H^2} \lesssim \tau^{-1} \sup_{t' \in \{t - \tau, t - \tau/2\}} \|R(t')\|_{H^2}.
$$

Iterating this estimate as many times as needed, in steps of size $\tau/2$, yields

$$
(10.6) \quad \sup_{t \in \{t_0, t_*\}} \|R(t)\|_{H^2} \leq C,
$$

where $C$ is a positive constant depending only on $Q_0, K_0, |\Sigma_0|, t_*, I_0$ and the initial data $\|R(t_0)\|_{H^2}$.

Now we are ready to show that the quantity $R_*$ defined by (10.1) is uniformly bounded for all $t_0 \leq t < t_*$. Although the argument is standard, we will include the details for completeness.

We have defined in (2.5) the electric and magnetic parts $E, H$ of the curvature tensor $R$. It is known that

$$
(10.7) \quad \nabla_i k_{jm} - \nabla_j k_{im} = \epsilon_{ij}^l H_{lm},
$$

$$
(10.8) \quad R_{ij} - k_{ia} k^{aj} + \text{Tr} k_{ij} = E_{ij}.
$$

From Lemma [2.1] and Lemma [2.2] it follows that

$$
(10.9) \quad \|\text{Ric}\|_{L^2} + \|k\|_{H^1} + \|E\|_{L^2} + \|H\|_{L^2} \leq C,
$$

where and in the following all the norms are taken over a fixed slice $\Sigma_t$ which is suppressed for simplicity.

In order to obtain the derivative estimates, by straightforward calculation we have symbolically

$$
(10.10) \quad \nabla_m E_{ij} = D_m R_{0i0j} - k \cdot H,
$$

$$
(10.11) \quad \nabla_m H_{ij} = D_m * R_{0i0j} - k \cdot E,
$$

$$
(10.12) \quad \nabla_m E_{ij} = D_m R_{0i0j} - k_{mn} D_0 R_{0m0j} - \nabla (k \cdot H),
$$

$$
(10.13) \quad \nabla_m H_{ij} = D_m * R_{0i0j} - k_{mn} D_0 * R_{0m0j} - \nabla (k \cdot E).
$$
From (10.10) and (10.11) it follows that
\[ \| \nabla E \|_{L^2} \leq \| D R \|_{L^2} + \| k \|_{L^6} \| H \|_{L^3} \]
\[ \| \nabla H \|_{L^2} \leq \| D R \|_{L^2} + \| k \|_{L^6} \| E \|_{L^3} \]
Applying Lemma 2.3 to \( E \|_{L^3} \) and \( H \|_{L^3} \), and using (10.6) and (10.9), we obtain
\[ \| \nabla E \|_{L^2} + \| \nabla H \|_{L^2} \leq C + C \left( \| \nabla E \|_{L^2}^{1/2} + \| \nabla H \|_{L^2}^{1/2} \right) \]
which implies
\[ (10.14) \quad \| \nabla E \|_{L^2} + \| \nabla H \|_{L^2} \leq C. \]

Next we will derive the estimate for \( \| \nabla^2 k \|_{L^2} \). It follows from \( \text{div} k = 0 \) and (10.7) that \( \Delta k = R \cdot k + \nabla H \). Differentiating it and commuting \( \nabla \) with \( \Delta \) gives
\[ \Delta \nabla k = R \cdot \nabla k + \nabla R \cdot k + \nabla^2 H \]
which together with (10.8) implies
\[ (10.15) \quad \Delta \nabla k = k \cdot k \cdot \nabla k + E \cdot \nabla k + \nabla E \cdot k + \nabla^2 H. \]

Multiplying (10.15) by \( \nabla k \) and integrating over \( \Sigma_t \) yields
\[
\int_{\Sigma_t} |\nabla^2 k|^2 \leq \int_{\Sigma_t} \left( |k|^2 |\nabla k|^2 + |E| |\nabla k|^2 + |\nabla E||k||\nabla k| + |\nabla H||\nabla^2 k| \right)
\leq \|k\|^2_{L^6} \|\nabla k\|^2_{L^{12/5}} + \|E\|_{L^6} \|\nabla k\|^2_{L^{12/5}} + \|\nabla E\|_{L^2} \|\nabla k\|_{L^3} \|k\|_{L^6}
+ \|\nabla H\|_{L^2} \|\nabla^2 k\|_{L^2}.
\]
With the help of Lemma 2.3 (10.3) and (10.4), we have
\[ \| \nabla^2 k \|_{L^2}^2 \leq C \left( \| \nabla^2 k \|_{L^2} + \| \nabla^2 k \|_{L^2}^{1/2} \right) \]
which implies \( \| \nabla^2 k \|_{L^2} \leq C \). By the Sobolev embedding we obtain
\[ (10.16) \quad \| k \|_{L^\infty} + \| k \|_{H^2} \leq C. \]

Using (10.16) and (10.6), it follows easily from (10.8), (10.12) and (10.13) that
\[ \| \nabla R \|_{L^2} + \| \nabla^2 R \|_{L^2} + \| \nabla^2 E \|_{L^2} + \| \nabla^2 H \|_{L^2} \leq C. \]

Finally we derive the estimate on \( \| \nabla^3 k \|_{L^2} \). By differentiating (10.15), commuting \( \nabla \) with \( \Delta \) and using (10.8) we obtain
\[ \Delta \nabla^2 k = k \cdot k \cdot \nabla^2 k + k \cdot \nabla k \cdot \nabla k + E \cdot \nabla^2 k + \nabla E \cdot k + \nabla^2 E \cdot k + k \cdot \nabla^3 H. \]
Multiplying this equation by \( \nabla^2 k \) and integrating over \( \Sigma_t \) it follows
\[
\int_{\Sigma_t} |\nabla^3 k|^2 \leq \int_{\Sigma_t} \left( |k|^2 |\nabla^2 k|^2 + |k||\nabla k|^2 |\nabla^2 k| + |E| |\nabla^2 k|^2 + |\nabla E||\nabla k||\nabla^2 k| \right)
+ \int_{\Sigma_t} \left( |k||\nabla^2 E||\nabla^2 k| + |\nabla^2 H||\nabla^3 k| \right)
\leq \|k\|^2_{L^\infty} \|\nabla^2 k\|^2_{L^2} + \|k\|_{L^\infty} \|\nabla^2 k\|^2_{L^2} + \|E\|_{L^\infty} \|\nabla^2 k\|^2_{L^2}
+ \|\nabla E\|_{L^2} \|\nabla k\|_{L^4} \|\nabla^2 k\|_{L^2} + \|k\|_{L^\infty} \|\nabla^2 E\|_{L^2} \|\nabla^2 k\|_{L^2} + \|\nabla^2 H\|_{L^2} \|\nabla^3 k\|_{L^2}
\leq C + C \|\nabla^3 k\|_{L^2}.
\]
Therefore \( \| \nabla^3 k \|_{L^2} \leq C \). The proof is thus complete.
11. Appendix

In this appendix we give the proof of Lemma 7.3. It suffices to consider the case that $F$ is an arbitrary smooth function on $\Sigma_t$.

On $\text{Int} S_{t,u}(t) = \bigcup_{u_m(t) \leq u \leq u_M(t)} S_{t,u}$, we have the family of diffeomorphisms

$$\Psi_{u,t} : S^2 \to S_{t,u}, \quad \Psi_{u,t}(\omega) = G_{\Phi(u_M(t))}(t,\omega),$$

where $G$ is defined in (1.13). Relative to this radial foliation, the metric $g$ on $\text{Int} S_{t,u}$ can be written as

$$a^2 du^2 + \gamma_{AB} d\omega^A d\omega^B,$$

where $\gamma$ is the restriction of $g$ on $S_{t,u}$. By Lemma 7.1, $\gamma_{S_{t,u}} \approx r(t,u)^2 \gamma_{S^2}$. Moreover, $F(x), x \in S_{t,u}$, can be reparametrized by

$$F(x) = F(u,\omega) := F \circ \Psi_{u,t}(\omega), \quad \omega \in S^2.$$

Due to Lemma 7.1 and (7.4), for a scalar function $f$,

$$(11.1) \quad \|f\|_{L^2(S_{t,u})} \approx \int_{S^2} |f|^2 (u - u_m)^2 d\mu_{S^2}.$$

For a fixed leaf $S_{t,u_0}$ with $u_m(t) < u_0 \leq u_M(t)$ and any $x \in S_{t,u_0}$, $F(x) = F(u_0,\omega)$ with $\omega \in S^2$, we define with $u_m := u_m(t)$

$$m(x) := \frac{2}{u_0 - u_m} \int_0^{u_0 - u_m} F(-z + u_0,\omega) dz$$

$$G(x) := m(x) - F(x).$$

Lemma 7.3 can be proved by establishing the following estimates

$$(11.2) \quad r^{-1/2} \|G\|_{L^2(S_{t,u_0})} \lesssim \|\nabla F\|_{L^2(\Sigma_t)}$$

$$(11.3) \quad r^{-1/2} \|m\|_{L^2(S_{t,u_0})} \lesssim \|F\|_{H^1(\Sigma_t)},$$

where $r = r(t,u_0)$.

To see (11.2), according to definition, we have

$$(11.4) \quad G(x) = \frac{2}{u_0 - u_m} \int_0^{u_0 - u_m} \int_0^1 \frac{d}{d\ell} F(u_0 - \ell z,\omega) d\ell dz.$$

It is easy to see

$$\frac{d}{d\ell} F(u_0 - \ell z,\omega) = -z \cdot \partial_u F(u_0 - \ell z,\omega).$$

In view of (BA1) and $N = -a^{-1} \partial_u$, it follows that

$$(11.5) \quad \left| \frac{d}{d\ell} F(u_0 - \ell z,\omega) \right| \lesssim |\nabla_N F|(u_0 - \ell z,\omega).$$

Since $0 < z \leq \frac{u_0 - u_m}{2}$, we have from (7.4) that $z \lesssim r'$, where $r' = r(t,u_0 - \ell z)$. Thus, by combining (11.4) with (11.5) and setting $v(y) := \|r' \nabla F(-y + u_0,\cdot)\|_{L^2}$, it
yields
\[
\begin{align*}
    r^{-1/2} \|G\|_{L^2(S^{i},\omega)} & \lesssim r^{-1/2} \int_0^1 \frac{1}{\ell} \int_0^{\frac{u_0-u_m}{2}} r'|\nabla F(-\ell z + u_0,\cdot)\|_{L^2} dz \\
    & \lesssim r^{-1/2} \int_0^1 \frac{1}{\ell} \int_0^{\frac{u_0-u_m}{2}} v(y)dyd\ell \\
    & \lesssim r^{-1/2} \int_0^1 \frac{u_0-u_m}{2} \int_0^{\frac{u_0-u_m}{2}} \ell \frac{1}{2} \int d\ell v(y)dy \\
    & \lesssim (u_0-u_m)^{-1/2} \int_0^{\frac{u_0-u_m}{2}} \ln \left( \frac{u_0-u_m}{2y} \right) v(y)dy.
\end{align*}
\]
By Hölder inequality,
\[
\begin{align*}
    r^{-1/2} \|G\|_{L^2(S^{i},\omega)} & \lesssim \left( \int_0^1 (\ln \sigma)^2 d\sigma \right)^{1/2} \left( \int_0^{\frac{u_0-u_m}{2}} |v(y)|^2 dy \right)^{1/2} \\
    & \lesssim \|\nabla F\|_{L^2(\text{Int}S^{i},\omega)}.
\end{align*}
\]
This proves (11.2). Using (11.3), with \( r'' := r(t, u_0 - z) \approx u_0 - u_m - z \),
\[
\|m\|_{L^2(S^{i},\omega)} \lesssim \int_0^{\frac{u_0-u_m}{2}} \|r''^{1/3} F\|_{L^6} r''^{-\frac{1}{2}} du' \lesssim r^{1/2} \|F\|_{L^6(\text{Int}S^{i},\omega)}.
\]
By Sobolev embedding, (11.3) follows.

References


Department of Mathematics, Stony Brook University, Stony Brook, NY 11794
E-mail address: qwang@math.sunysb.edu