Diffeomorphisms in group field theories

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(Received 12 February 2011; published 27 May 2011)

We study the issue of diffeomorphism symmetry in group field theories (GFT), using the non-commutative metric representation introduced by A. Baratin and D. Oriti [Phys. Rev. Lett. 105, 221302 (2010)]. In the colored Boulavat model for 3d gravity, we identify a field (quantum) symmetry which ties together the vertex translation invariance of discrete gravity, the flatness constraint of canonical quantum gravity, and the topological (coarse-graining) identities for the 6j symbols. We also show how, for the GFT graphs dual to manifolds, the invariance of the Feynman amplitudes encodes the discrete residual action of diffeomorphisms in simplicial gravity path integrals. We extend the results to GFT models for higher-dimensional BF theories and discuss various insights that they provide on the GFT formalism itself.

DOI: 10.1103/PhysRevD.83.104051 PACS numbers: 04.60.Pp, 02.40.Gh, 04.60.Gw, 04.60.Nc

I. INTRODUCTION

Diffeomorphism symmetry is a crucial aspect of the dynamics of spacetime geometry as described by general relativity and its higher derivative extensions. It is tied to the notion of background independence [1], as the introduction of a nondynamical background breaks the full diffeomorphism invariance. It also imposes strong constraints on the allowed dynamics. In fact, for example, the only diffeomorphism invariant action (in 4d) for a tensor metric field that involves, at most, its first derivatives is the Einstein-Hilbert action (with cosmological constant); and, in a canonical formalism based on intrinsic metric and conjugate extrinsic curvature, only canonical general relativity is compatible with the algebra of (the canonical counterpart of) diffeomorphisms [2].

This fact acquires even more relevance from the point of view of ongoing efforts to build a quantum theory of gravity. In background independent approaches [3] aiming at explaining the very origin of spacetime geometry, starting from “pregeometric,” discrete, or purely algebraic structures, the correct implementation of diffeomorphism invariance is a key guiding principle for the very definition of the microscopic dynamics. A major open problem in these approaches, such as in simplicial gravity [4], spin foam models [5], and group field theories (GFT) [6], is to show how the dynamics reduce to general relativity in a semiclassical and continuum approximation. A good control over the (pregeometric analogue of) diffeomorphism invariance is then essential: provided such an approximation does not break this symmetry, general relativity should emerge as the dynamics of the metric field defined in terms of the fundamental degrees of freedom of the theory, at least at leading order. If the invariance is only approximate, still the requirement that it becomes exact in the continuum limit is an important guiding principle for the definition of appropriate coarse-graining and renormalization procedures, or to identify the diffeomorphism invariant sector which should be dominant in the limit [7].

With the smooth manifold of general relativity replaced by discrete structures, the issue becomes that of identifying suitable transformations of the pregeometric data, leaving the quantum amplitudes invariant, and encoding the (residual) action of the diffeomorphism group. This is known in the context of Regge calculus [9], where an action of diffeomorphisms at the vertices of the Regge triangulation has been shown to exist around flat solutions. This is understood geometrically as the invariance of the Regge action upon translations of the vertices, in a local flat embedding of the triangulation in \( \mathbb{R}^d \). The invariance is exact in 3d, where the geometry is constrained to be flat; it is only approximate in the 4d case and in the presence of a cosmological constant (see [10] and references therein). In both cases, the (approximate) invariance can be related to discrete Bianchi identities. The action of diffeomorphisms in spin foam models has also been studied in the context of 3d gravity [11]. In this work, it is shown that the discrete residual of the local Poincaré invariance, classically equivalent to diffeomorphism invariance, is responsible for (part of) the divergences of the Ponzano-Regge model. A related aspect of diffeomorphisms in spin foam models is the algebraic expression of diffeomorphism invariance in terms of algebraic identities satisfied by \( n-j \) symbols, at

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1In dynamical triangulations [4,8], all such data are fixed to constant values, and the only analogue of diffeos is the automorphism group of the simplicial complex itself.
the root of the topological invariance of some models, and
recognized to be an algebraic translation of the canonical
gravity constraints [12,13].

GFTs [6] are a higher-dimensional generalization of
matrix models [14] and provide a second quantization of
both spin network dynamics and simplicial gravity. Their
Feynman diagrams are dual to simplicial complexes; the
amplitudes are given equivalently as spin foam models or
simplicial gravity path integrals [15]. Conversely, any spin
foam model can be interpreted as a Feynman amplitude of
simplicial gravity path integrals [15]. Hence, in the GFT
perturbative expansion, one obtains a sum over (pre)geometric data
weighted by appropriate amplitudes, augmented by a
sum over simplicial complexes of arbitrary topology. In
this paper, we ask ourselves whether the various notions of
diffeomorphism invariance studied in the literature on dis-
crete gravity can be traced back to a symmetry of the group
field theory.

This task had proven impossible to fulfill up to now. The
main reason was the absence, at the GFT level, of explicit
metric variables, on which (discrete) diffeomorphisms
would act. Now, recently, a metric formulation of GFT,
completely equivalent to the usual formulations in terms of
group variables or group representations, has been de-
veloped [15] and used to prove an exact duality between
spin foam models and simplicial path integrals. Here, we
use this formulation to study the action of discrete diffeo-
morphisms in GFT. By doing so, we relate in a clear way
various aspects of diffeomorphism invariance in spin foam
models, canonical loop quantum gravity, and simplicial
gravity. More precisely, we show that there is a set of field
transformations leaving the GFT action invariant, whose
geometrical meaning in the various GFT representations
ties together the symmetry of the Regge action and the
simplicial Bianchi identities, the canonical constraints of
loop quantum gravity (adapted to a simplicial complex),
and algebraic identities satisfied by $n - j$ symbols.

A key feature of this metric formulation, which recasts
GFTs as noncommutative field theories on Lie algebras, is
to reveal and to make explicit the noncommutativity of the
geometry in GFT and spin foam models [17–19]. The action
of discrete diffeomorphisms described in this paper natu-
really incorporates this noncommutativity, as it is generated
by a Hopf algebra [20]. Diffeomorphism invariance in GFT
thus takes the form of a deformed (quantum) symmetry. The
definition of deformed symmetries in GFT, also considered
in [21], requires to embed the field theory into the larger
framework of braided quantum field theories [22].

We work in the colored version of the GFT formalism
[23,24], analogous to multimatrix models [14]. The coloring
can be used [25] to define a full homology for the GFT
colored diagrams\(^2\) and to unambiguously associate to it a

\(^2\)For alternative definitions of homology of GFT diagrams,
see [26].

triangulated pseudomanifold, that is, complexes with point-
like topological singularities [27]. The color formalism
eliminates more pathological diagrams that are instead
generated by standard GFTs [23]. Strikingly, the coloring
turns out to be also crucial for recasting the perturbative
expansion of the (colored) Boulatov model, with a cutoff in
representation space, in terms of a topological expansion,
and to show that the sum is dominated by manifolds of
trivial topology in the large cutoff limit [28]. This is the GFT
analogue of the “large-N” expansion of matrix models.
These are very strong motivations for introducing coloring
in GFT models. In this paper, we give another one: it is only
in the colored framework that the action of discrete diffeo-
morphisms can be encoded into field transformations.

We focus on the topological models—namely, the (col-
ored) Boulatov and Ooguri models for \(3d\) gravity and
\(4d\ BF\) theory. The analysis can, however, be extended to
\(4d\) gravity models obtained by imposing constraints on
topological ones [29].

The paper is organized as follows. In Sec. II, we review
the GFT framework in dimension three, in its three known
formulations: the “group” formulation in terms of fields
on a group manifold, the “spin” formulation in terms of
tensors in group representations, and the recent “metric”
formulation in terms of fields on Lie algebras. We illustrate
how the duality of GFT representations translates into an
exact duality between spin foam models, lattice gauge
theory, and simplicial path integrals.

In Sec. III, we introduce a set of field transformations
which, we show, leaves invariant the action of the colored
Boulatov for \(3d\) gravity. These transformations are gener-
ated by a Hopf algebra [20], more precisely by the trans-
lational part of a deformation of the Poincaré group. The
definition of deformed (quantum) symmetries on GFT
requires to embed the field theory into the larger frame-
work of braided quantum field theories [22]. We exploit the
invariance of the GFT vertex function to give the geomet-
rical meaning of the symmetry in the three GFT represen-
tations. We find that:

1. in the “metric” representation, the symmetry re-
ffects the invariance under translations of each of
the vertices of the Euclidean tetrahedron patterned
by the GFT interaction.

2. in the “group” representation, the symmetry ex-
presses the flatness of the boundary connection
that the field variables represent.

3. in the “spin” representation, the symmetry encodes
the topological identities and recursion relations of
the \(6j\) symbols.

In Sec. IV, we look at the invariance of the GFT ampli-
tudes and explain how the GFT symmetry relates to the
action of diffeomorphisms in simplicial path integrals. The
analysis naturally distinguishes between manifold graphs
and pseudomanifold ones. In the case of manifold graphs,
we show, both geometrically and algebraically, how to
derive discrete Bianchi identities from the invariance of the vertex and propagator functions.

Finally, in Sec. V, we extend the results to the GFT model for 4d BF theory and discuss the case of constrained models for gravity. We conclude in Sec. VI with a discussion of various issues raised by our analysis and new insights that it provides on the GFT formalism.

II. COLORED GFTs AND METRIC REPRESENTATION

d-dimensional GFTs [6], in their colored version [23], are field theories described in terms of \( d + 1 \) complex fields \( \{ \varphi_\ell \}_{\ell=1}^{d+1} \) defined over \( d \) copies of a group \( G \), with a certain gauge invariance. The index \( \ell \) is referred to as the color of the fields. Here, we consider the 3d case and the Euclidean rotation group \( G = \text{SO}(3) \), so that each field \( \varphi_\ell \) is a function on \( \text{SO}(3) \).

The gauge invariance condition reads:

\[
\forall h \in \text{SO}(3), \quad \varphi_\ell(hg_1, hg_2, hg_3) = \varphi_\ell(g_1, g_2, g_3).
\]  

(1)

The dynamics is governed by the action \( S[\varphi] = S_{\text{kin}}[\varphi] + S_{\text{int}}[\varphi] \), where the kinetic term couples fields with the same colors:

\[
S_{\text{kin}}[\varphi] = \int [dg_1]_1 [dg_2]_2 [dg_3]_3 \sum_{\ell=1}^4 \varphi_\ell(g_1, g_2, g_3) \varphi_\ell^*(g_1, g_2, g_3),
\]  

(2)

where \( [dg]_n \) is the product Haar measure on the group \( \text{SO}(3)^n \), and \( \varphi_\ell^* \) are the complex conjugated fields. The interaction is homogeneous of degree four and given by

\[
S_{\text{int}}[\varphi] \propto \int [dg_1]_1 [dg_2]_2 [dg_3]_3 \varphi_1(g_1, g_2, g_3) \varphi_2(g_3, g_4, g_5) \\
\quad \times \varphi_3(g_5, g_2, g_6) \varphi_4(g_6, g_4, g_1) \\
+ \lambda \int [dg_1]_1 [dg_2]_2 [dg_3]_3 [dg_4]_4 \varphi_1(g_1, g_2, g_3) \varphi_2(g_3, g_4, g_5) \\
\quad \times \varphi_3(g_5, g_4, g_6) \varphi_4(g_6, g_2, g_1).
\]  

(3)

The six integration variables in each integral follow the pattern of the edges of a tetrahedron. A field represents a triangle, the three field arguments being associated to its edges (see Fig. 1). The four triangles of the tetrahedron are marked by distinct colors.\(^3\) When the fields with different colors are all identified \( \varphi_\ell := \varphi \) to a single real field, colored GFTs reduce to standard GFTs.

The Feynman expansion of a GFT generates stranded diagrams, with three strands per propagator, equipped with a canonical orientation of all lines and higher-dimensional faces. The propagator and vertex for \( \varphi \) are drawn in Fig. 2; the vertex for \( \varphi \) is obtained by reversing the order of all labels. While the interaction vertex patterns a tetrahedron with colored triangles, the propagator glues together tetrahedra along triangles of the same color.

Graph amplitudes are built out of propagators and vertex functions:

\[
P_\ell(g, g') = \int dh \prod_{i=1}^3 \delta(g_i^{-1} h g_i'),
\]  

(4)

\[
V(g, g') = \int dh_\ell \prod_{i=1}^6 \delta((g_i^{-1})^{-1} h_\ell h_\ell^{-1} g_i^\ell),
\]

which identifies the variables along connected strands, modulo left shift by the gauge variables \( h \) arising from the invariance \( (1) \). The vertex function has an interpretation in terms of lattice gauge theory, where the three group variables \( g_i^\ell \) and the group variables \( h_\ell \) are viewed as holonomies along the links of the complex topologically dual to a tetrahedron, shown in Fig. 3. The \( g_i^\ell \) are “boundary” holonomies along the links dual to a triangle \( \ell \). The \( h_\ell \) are “bulk” holonomies along the links connecting the triangles to the center of the tetrahedron. The vertex function simply states that the two-dimensional faces of the complex dual [in red (light gray) in Fig. 3] are flat. This implies that the encoding of geometric information in the model fits a piecewise-flat context, as in simplicial quantum gravity approaches.

\(^3\)The coloring of each field, and thus of each triangle, by a single label \( \ell \) can be equivalently converted in a coloring of each vertex of the tetrahedron by a label in the same range. In this setting, each field triangle is labeled by the three colors of its three vertices. This shows that colored GFTs are a field theory generalization of double-indexed 3d tensor models [30].
In gluing two tetrahedra, the propagator function identifies the boundary variables of the shared triangle, up to a group variable $h$ interpreted as a further parallel transport through the triangle.

After integration over all boundary variables $g$, the amplitude of a closed GFT diagram $G$ takes the form

$$\mathcal{A}_G = \int \prod_f \! dh \! \prod_f \! \delta \left( \prod_{i \in f} h_i \right),$$

where the products are over the lines $l$ and the faces (loops of strands) $f$ of the diagram. $l$ and $f$ dually label the triangles and the edges of the triangulation defined by the diagram. The ordered products of line variables along the boundary $\partial f$ of the faces are computed by choosing an orientation and a reference vertex for each face $f$. The group variables are taken to be $h_i$ or $h_i^{-1}$, depending on whether the orientations of $l$ and $f$ agree or not. In terms of lattice gauge theory, the set of variables $(h_i)_{i \in G}$ gives a discrete connection on (the complex dual to) the triangulation, giving parallel transports from one tetrahedron to another. The delta functions in (5) impose this connection to be flat. The model is already seen then as describing a discrete version of topological 3d BF theory, discretized on the simplicial complex dual to the GFT diagram.

The “spin” representation of the GFT is obtained using the Peter-Weyl expansion of the fields over half-integer spins labeling the representations of $SO(3)$. Because of gauge invariance, the coefficients are proportional to the $SO(3)$ Clebsh-Gordan coefficients $C_{j_1,j_2,j_3}^{j_0,m_1,m_2,m_3};$ the interaction vertex is expressed in terms of $6j$ symbols. The amplitude of a Feynman diagram gives the Ponzano-Regge spin foam model [31]:

$$\mathcal{A}_G = \sum_{j_0,j_1} \prod_{j \in e} \! \prod_{\tau} \! \left( \prod_{i \in \delta \tau} \! j_i^\tau \right)^{2j+1},$$

where the spins $j_\tau$ label the edges of the triangulation associated to the diagram, $d_j = 2j + 1$ is the dimension of the representation $j$, and the amplitude is a product of tetrahedral $6j$ symbols. Thus, group and spin representations of the GFT realize explicitly the duality between the connection (5) and spin foam (6) formulations of the Ponzano-Regge model [5,32].

A third representation of GFTs, in terms of continuous noncommutative “metric” variables $x \in \mathfrak{su}(2) \sim \mathbb{R}^3$, has been recently developed [15] and shown to realize a further duality between spin models and simplicial path integrals. Since the geometrical meaning of the symmetries studied in the next sections is best understood in such a metric representation, let us briefly recall here its construction. The representation is obtained using the group Fourier transform [17,20] of the fields

$$\hat{\varphi}_\ell(x_1,x_2,x_3) := \int [dg] \! \varphi_\ell(g_1,g_2,g_3)e_{g_1}(x_1)e_{g_2}(x_2)e_{g_3}(x_3),$$

expressed in terms of plane-wave functions $e_g : \mathfrak{su}(2) \sim \mathbb{R}^3 \to \mathbb{C}$.

The image of the Fourier transform inherits by duality a nontrivial (noncommutative) pointwise product from the convolution product on the group. It is defined on plane waves as

$$(e_g \ast e_{g'})(x) := e_{gg'}(x),$$

and extends componentwise to the product of three plane waves and by linearity to the whole image of the Fourier transform.

The first feature of this representation is that the gauge invariance condition (1) expresses itself as a “closure constraint” for the triple of variables $x_i$ of the dual field. To see this, we consider the projector $\mathcal{P}$ onto gauge invariant fields:

$$\mathcal{P} \varphi_\ell = \int [dh] \! \varphi_\ell(hg_1,hg_2,hg_3),$$

and note that

$$\widehat{\mathcal{P} \varphi_\ell} = \hat{\mathcal{C}} \ast \hat{\varphi}_\ell, \quad \hat{\mathcal{C}}(x_1,x_2,x_3) := \delta_0(x_1 + x_2 + x_3),$$

where $\delta_0$ is the element $x = 0$ of the family of functions.
where $d^3 y$ is the standard Lebesgue measure on $\mathbb{R}^3$. We may thus interpret the variables $x_i$ of the dual gauge invariant field as the closed edge vectors of a triangle in $\mathbb{R}^3$ and further confirm the interpretation of the GFT fields as (noncommutative) triangles.

The GFT action can be written in terms of dual fields and metric variables by exploiting the duality between group convolution and $\star$ product. Given two functions $f, h$ on $\text{SO}(3)$ and $\hat{f}, \hat{h}$ their Fourier transform (7), this duality can be read in the property

$$\int [df] [g(h)g(h) = \int [d^3x] [\hat{f} \ast \hat{h}_{-}(x)],$$

where $\hat{h}_{-}(x) := \hat{h}(-x)$, and $d^3 x$ is the Lebesgue measure on $\mathbb{R}^3$. Hence, the combinatorial structure of the GFT action in the metric representation is the same as in group one, while group convolution is replaced by $\star$ product. Using the short notation $\hat{\phi}_{123} := \hat{\phi}(x_1, x_2, x_3)$, we can write the action as

$$S[\hat{\phi}] = \int [d^3x_i]^3 \sum_{\ell=1}^{4} \hat{\phi}_{123}^{\ell} \ast \hat{\phi}_{123}^{\ell} + \lambda \int [d^3x_i]^3 \hat{\phi}_{123}^{345} \ast \hat{\phi}_{345}^{526} \ast \hat{\phi}_{4}^{526} \ast \hat{\phi}_{4}^{641} + \lambda \int [d^3x_i]^3 \hat{\phi}_{4}^{146} \ast \hat{\phi}_{345}^{526} \ast \hat{\phi}_{2}^{526} \ast \hat{\phi}_{123}^{641},$$

where it is understood that $\star$ products relate repeated upper indices as $\hat{\phi}^{\ell} \star \hat{\phi}^{\ell} := (\hat{\phi} \ast \hat{\phi}_{-})(x_\ell)$, with $\hat{\phi}_{-}(x) := \hat{\phi}(-x)$.

Feynman amplitudes are built out of propagators and vertex functions:

$$P_{\ell}(x, x') = \int [dh] [\delta_{\ell} - \ast e_h(x)],$$

$$V(x, x') = \int [d^3 h] [\delta_{\ell} - \ast e_h, e_h^{-1}](x),$$

where the $\delta_{\ell}$ are given by (13). These have a natural interpretation in terms of simplicial geometry, where the $x$ variables on connected strands encode the metric of the same edge in different frames, related with each other by the holonomies $h$. In building up the diagram, propagator and vertex strands are joined to one another using the $\star$ product.

Under the integration over the holonomy variables, the amplitude of a closed diagram $G$ factorizes into a product of face amplitudes $A_f[h]$, taking the form of a cyclic $\star$ product:

$$A_f[h] = \prod_{j=0}^{N_f} [d^3x_j] \prod_{j=0}^{N_f} (\delta_{x_j} \ast e_{h_{j+1}})(x_{j+1})$$

where the product is over the $N_f$ vertices of $G$ (dual to tetrahedra) in the loop of strands that bound $f$. The ordered $\star$ product is computed by choosing an orientation and a reference vertex for the face $f$; by convention, we set $x_{N_f+1} := x_0$. The holonomy $h_{j+1}$ parallel transports the reference frame of $j$ to that of $j + 1$. In terms of simplicial geometry, it encodes the identification, up to parallel transport, of the metric variables associated to the edge dual to $f$ in the different frames $f$.

After integration, within all face amplitudes, over all metric variables $x_j$ except for that $x_0$ of the reference frame, the amplitude of the GFT diagram $G$ takes the form of a simplicial path integral:

$$A_G = \int [d^3h] [d^3x_e] e^\sum_{e \in E} \text{Tr}_{H_e}$$

where the products are over the lines of $G$ and the edges of the dual triangulation, and $H_e := \prod_{i \in e} h_i$ is the holonomy along the boundary of the face $f_e$ of $G$ dual to $e$, calculated from a given reference tetrahedron frame. The exponential term is the (exponential of the) discrete action of first-order $3d$ gravity (which is the same as $3d$ BF theory), in Euclidean signature. This gives the definite confirmation of the interpretation of the $x_e$ variables as discrete triad variables associated to the edges of the triangulation dual to the GFT Feynman diagram (edge vectors).

Thus, the metric representation of GFT realizes explicitly the duality between spin foam models (5) or (6) and simplicial path integrals (19), generalizing it to arbitrary transition amplitudes (corresponding to open GFT diagrams) with appropriate boundary terms arising naturally in the simplicial action in (19), for fixed triad variables at the boundary, and boundary observables. This result is general: it extends to BF theories in higher dimensions and to gravity models obtained as constrained BF theories (see [15,33]).

The metric representation has, of course, the advantage of making the (noncommutative, simplicial) geometry of GFT and spin foam models more transparent. This will be useful for the understanding of the symmetries studied in the next section.

### III. GFT (DISCRETE) DIFFEOMORPHISM SYMMETRY

In this section, we introduce a set of field transformations which, we show, leave the GFT action invariant. We give the geometrical meaning of such transformations in...
the different representations and show that they correspond to diffeomorphisms in discrete quantum geometry models. We also derive yet another GFT representation in terms of the generators of the symmetry, which are Lie algebra “position” variables associated to the vertices of the simplex patterned by the GFT field.

The noncommutativity of the metric (triad) space plays a crucial role in the definition and meaning of the symmetry transformation. This is, in fact, a Hopf algebra (quantum) symmetry, characterized by a nontrivial action on a tensor product of fields, due to a nontrivial coproduct. The relevant quantum group here, i.e., in this specific GFT model for Euclidean 3d gravity with the local gauge group being SO(3), is a deformation of the Euclidean group ISO(3), the so-called Drinfeld double DS\(3\).\(^4\)

### A. Action of DS\(3\) on fields on SO(3)

The Drinfeld double is defined as DS\(3\) = \(\mathcal{C}(\text{SO}(3)) \otimes \mathcal{C}(\text{SO}(3))\), where the group algebra \(\mathcal{C}(\text{SO}(3))\) acts by the adjoint action on the algebra of functions \(\mathcal{C}(\text{SO}(3))\). It is a deformation of the three-dimensional Euclidean group ISO(3)—more precisely, of the Hopf algebra \(\mathcal{C}(\mathbb{R}^3) \otimes \mathcal{C}(\text{SO}(3))\)—where the “rotations” belong to the group algebra \(\mathcal{C}(\text{SO}(3))\) and the “translations” are complex functions in \(\mathcal{C}(\text{SO}(3))\). A general element can be written as a linear combination of elements \(f \otimes \Lambda\), where \(f \in \mathcal{C}(\text{SO}(3))\) and \(\Lambda \in \text{SO}(3)\). The space of functions on the group \(\mathcal{C}(\text{SO}(3))\) gives a representation of DS\(3\), in which rotations act by adjoint action on the variable and translations act by multiplication:

\[
\phi(g) \mapsto \phi(\Lambda^{-1} g \Lambda) = \phi(\Lambda^{-1} g) \phi(g),
\]

\[
\phi \in \mathcal{C}(\text{SO}(3)).
\]

Choosing as a translation element a generating plane wave labeled by \(e \in \mathfrak{su}(2) \sim \mathbb{R}^3\), the field \(\phi\) gets multiplied by a phase \(f_x(g) = e_x(e)\). Upon the group Fourier transform introduced in the previous section,

\[
\hat{\phi}(x) = \int [dg] \phi(g) e_x(g),
\]

this corresponds to the dual action \(\hat{\phi}(x) \mapsto \hat{\phi}(x + e)\). We will also use the dual action of DS\(3\), where rotations act by inverse adjoint action and plane waves labeled by \(e\) act, by translation, by \(-e\).

Up to now, the transformations are the exact analogue of the usual Poincaré transformations on functions on flat space, here replaced by the algebra \(\mathfrak{su}(2)\), while momentum space is replaced by the group manifold SO(3). The deformation manifests itself as a nontrivial action on a tensor product of fields, due to the nontrivial coproduct on the translation algebra \(\mathcal{C}(\text{SO}(3))\). Thus,

\[
\phi_1(g_1) \phi_2(g_2) \mapsto \Delta f(g_1 \otimes g_2) \phi_1(g_1) \phi_2(g_2),
\]

where the coproduct \(\Delta\) is given by

\[
\Delta f(g_1 \otimes g_2) = f(1) f(2) = f(g_1 g_2),
\]

\(\forall f \in \mathcal{C}(\text{SO}(3)).\)

Using the fact that \(e_{g_2}(e) = (e_{g_1} \ast e_{g_2})(e)\), one can check that the dual action of the plane wave \(e_x(e)\) on a tensor product is obtained by translating each variable by \(e\) and by taking the \(\ast\) product of the resulting fields with respect to \(e\):

\[
\hat{\phi}_1(x_1) \hat{\phi}_2(x_2) \mapsto \hat{\phi}_1(x_1 + e) \ast_{x_2} \hat{\phi}_2(x_2 + e).
\]

This structure is what replaces the usual translation group \(\mathbb{R}^3\), and the deformation is consistent with the noncommutativity of the algebra of functions on \(\mathfrak{su}(2) \sim \mathbb{R}^3\) induced by the \(\ast\) product.

### B. GFT as a braided quantum field theory

In order to allow the Hopf algebra to act on the polynomials of fields defined by the GFT action, the idea is to embed the theory into the algebraic framework of braided quantum field theories defined by Oeckl [22]. In short, this consists of lifting all polynomials of fields to tensor products, in order to keep track of the ordering of the fields and field variables. Commuting fields or field variables require the use of braiding maps \(B_{12} : X_1 \otimes X_2 \rightarrow X_2 \otimes X_1\) between any two copies of the space of fields. The theory is defined perturbatively as a braided Feynman diagram expansion, using a braided Wick theorem [22]. In a trivial embedding of GFTs, where all fields commute, into the braided framework, the braiding maps are chosen to be the trivial flip maps:

\[
B_{12} : \mathcal{C}(\text{SO}(3)) \otimes \mathcal{C}(\text{SO}(3)) \rightarrow \mathcal{C}(\text{SO}(3)) \otimes \mathcal{C}(\text{SO}(3))
\]

\[
\times \phi(g_1) \otimes \phi(g_2) \mapsto \phi(g_2) \otimes \phi(g_1).
\]

We emphasize that such trivial embedding does not modify the theory.\(^5\) It, however, allows us to define Hopf algebra transformations on the GFT fields.

The possibility of using a nontrivial braiding between fields or field arguments is not employed in usual GFTs, so we will stick to the usual formalism in what follows. The choice of trivial braiding is often made also in the noncommutative geometry literature, even in the presence of quantum group symmetries. However, since, in general, the trivial braiding map does not intertwine the action of the quantum group symmetry, this choice leads to a breaking of the symmetry at the level of the \(n\)-point functions. In order to make the full theory symmetric, it is most natural to use the braiding of the (braided) category of

\(^4\)The role played by the Drinfeld double in spin foam and GFT models has been emphasized already, e.g., in [21,34,35].

\(^5\)In fact, in this setting, the braided Feynmanology is redundant, and the braided amplitudes coincide with the unbraided ones.
Diffeomorphisms in Group Field Theories

C. Symmetries of the GFT action

We have recalled above how $\mathcal{D}SO(3)$ acts on a function of a single variable. Here, we define a set of transformations of the GFT fields $\varphi_\ell$ under rotations and translations which leave the GFT action invariant. As we illustrate in the different GFT representations in the next subsection, the translational part of this action, interpreted as “vertex translations” in the simplex patterned by the field, will encode the action of discrete diffeomorphisms in GFT.

Let us first point out that the requirement of gauge covariance restricts the number of independent transformations that a field transformation $T$ can undergo. Such transformations is, indeed, well-defined on a gauge invariant field only if it commutes with the projector (11):

$$\mathcal{P} \triangleright (T \triangleright \varphi_\ell) = T \triangleright (\mathcal{P} \triangleright \varphi_\ell).$$

Thus, for instance, the only gauge covariant action of the rotations in $\mathcal{C}SO(3) \otimes \mathbb{C}$,

$$\varphi_\ell(g_1, g_2, g_3) \mapsto \varphi_\ell(\Lambda_{1,\ell}^{-1} \triangleright g_1, \Lambda_{2,\ell}^{-1} \triangleright g_2, \Lambda_{3,\ell}^{-1} \triangleright g_3),$$

is the diagonal one, $\Lambda_{i,\ell} := \Lambda_\ell$. In the metric representation, gauge covariance simply means that the transformation preserves the closure $\delta(x_i^1 + x_i^2 + x_i^3)$ of the triangle $\ell$. In the case of rotations, one can easily go one step further and check that the only field transformation that preserves the kinetic and interaction polynomials is generated by a single rotation $\Lambda_\ell := \Lambda$. In the metric representation, this is the only action of the rotations that respects the gluing of edge vectors of the tetrahedron patterned by the interaction.

Let us stress that this symmetry corresponds precisely to the invariance under local changes of frame in each tetrahedron and in each triangle that one expects in 3d simplicial gravity (see Sec. IV). We thus find such an invariance implemented as the well-known local gauge invariance in both the simplicial path integral and pure gauge theory formulation of the GFT Feynman amplitudes, as well as in their spin foam representation.

We now turn to the more interesting case of translations. We will define transformations generated by four $\text{su}(2)$-translation parameters $e_\ell$, where $\nu$ labels the four vertices of the interaction tetrahedron, diagrammatically represented by its dual diagram in Fig. 2. Each vertex of this tetrahedron is represented by a certain subgraph, which we call the “vertex graph” [25,39]: the vertex graph for the vertex $v_\ell$ opposite to the triangle of color $\ell$ is obtained by removing all the lines which contain strands of color $\ell$. The vertex graph of $v_3$ is pictured in Fig. 4: its three lines pattern the three edges 1, 3, 4 sharing $v_3$.

The vertices opposite to the triangles $\ell = 2, 3, 4$ are represented by identical (after anticlockwise rotation by $\pi/4, \pi/2$, and $3\pi/4$) diagrams, where 1, 3, 4 are replaced, respectively, by 3, 5, 1; by 5, 6, 4; and by 6, 1, 2 (in this order).

To define the action of a translation of the vertex $v_3$, we equip the lines of the vertex graph with an orientation, as drawn in Fig. 4. Using this convention, each line has an “incoming” and an “outgoing” external strand. A translation of $v_3$ generated by $e_3 \in \mathfrak{su}(2)$ acts nontrivially only on the strands of the vertex graph. In the metric representation, it shifts the corresponding variables $x_i^\ell$ by $\pm e_3$, whether the strand $i$ comes in or out of $\ell$:

$$x_i^\ell \mapsto x_i^\ell \pm e_3$$

in a way that preserves the closure $\delta(x_1^\ell + x_2^\ell + x_3^\ell)$ of each triangle $\ell$. More precisely, the translation $T_{e_3}$ of the vertex $v_3$ acts on the dual fields as

$$T_{e_3} \triangleright \varphi_1(x_1, x_2, x_3) := \star_{e_3} \varphi_1(x_1 - e_3, x_2, x_3 + e_3),$$

$$T_{e_3} \triangleright \varphi_2(x_3, x_4, x_5) := \star_{e_3} \varphi_2(x_3 - e_3, x_4 + e_3, x_5),$$

$$T_{e_3} \triangleright \varphi_3(x_6, x_7, x_8) := \star_{e_3} \varphi_3(x_6, x_7 - e_3, x_8 + e_3),$$

$$T_{e_3} \triangleright \varphi_4(x_9, x_{10}, x_{11}) := \star_{e_3} \varphi_4(x_9, x_{10} - e_3, x_{11} + e_3).$$

FIG. 4. Vertex graph for the vertex $v_3$. 

$\text{Diffeomorphisms in Group Field Theories}$

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$$T_{e_3} \triangleright \varphi_4(x_9, x_{10}, x_{11}) := \star_{e_3} \varphi_4(x_9, x_{10} - e_3, x_{11} + e_3).$$

$\text{FIG. 4. Vertex graph for the vertex } v_3.$
The same field transformation can be expressed in a more explicit way (without star product) in the group representation, by group Fourier transform, as follows:

\[
\begin{align*}
T_{e_3} \phi_1(g_1, g_2, g_3) &:= e_{g_1^{-1}g_3}(e_3) \phi_1(g_1, g_2, g_3), \\
T_{e_3} \phi_2(g_3, g_4, g_5) &:= e_{g_3^{-1}g_5}(e_3) \phi_2(g_3, g_4, g_5), \\
T_{e_3} \phi_4(g_6, g_4, g_1) &:= e_{g_6^{-1}g_4}(e_3) \phi_4(g_6, g_4, g_1), \\
T_{e_3} \phi_3(g_5, g_2, g_6) &:= \phi_3(g_5, g_2, g_6).
\end{align*}
\]

(29)

The transformation is immediately extended to the complex conjugated fields \( \bar{\phi}_e \) by requiring consistency with complex conjugacy, using the property of the plane waves that \( \overline{e_g(e)} = e_{g^{-1}}(e) \).

We see at first glance in (28) the geometric meaning of this transformation as a vertex translation, by the way it affects the arguments of the dual fields interpreted as edge vectors. When translating a vertex of the triangle, one translates the two edges sharing this vertex; each edge is translated in an opposite way due to the orientation of the vertex. This transformation is geometrically invariant. Geometrically, this corresponds to the rather trivial fact that the geometry of a Euclidean triangle is invariant under a global translation of its vertices. Such a global translation is defined by choosing an order for the vertices of each triangle. For example, choosing the order \( v_3, v_4, v_2 \) for the vertices of the triangle \( \ell = 1 \), a global translation acts on \( \phi_1 \) as

\[
\phi_1(g_1, g_2, g_3) \mapsto e_{g_1^{-1}g_3}(e) \times e_{g_2^{-1}g_3}(e) \times e_{g_3^{-1}g_3}(e) \times \phi_1(g_1, g_2, g_3).
\]

Before discussing further the meaning of the symmetry in the next section, let us point out that the four symmetry generators are not all independent—in other words, the symmetry is reducible. In fact, there is a global translation of the four vertices of the tetrahedron under which the fields transform trivially. Geometrically, this corresponds to the rather trivial fact that the geometry of a Euclidean tetrahedron is invariant under a global translation of its vertices. Such a global translation is defined by choosing an order for the vertices of each triangle. For example, choosing the order \( v_3, v_4, v_2, v_1 \) for the vertices of the tetrahedron \( \ell = 1 \), a global translation acts on \( \phi_1 \) as


D. Invariance of the vertex and diffeomorphisms

We now want to show how the field symmetry (29), and, more specifically, the invariance of the vertex function, tie together various notions of (discrete residual of) diffeomorphisms studied in the literature. To do so, we probe the meaning of such invariance in the different GFT representations. This picture will be completed in the next section, when we will discuss the invariance of the GFT Feynman amplitudes and \( n \)-point functions.

(i) Metric representation. The vertex function is given by the formula (17)

\[
V(x_i^f, x_j^f) = \int \prod_{\ell=1}^{4} [dh_{\ell}] \prod_{i=1}^{6} \delta(x_i^f - h_{\ell} h_{\ell}^{-1} x_i^0). 
\]

(31)

Fixing the ordering of the variables to the one defined by the interaction polynomials, this function can be lifted to the group Fourier dual of a tensor product in \( \mathbb{C}(SO(3)^{\otimes 12}, \) invariant under the (non-commutative) translation (27). As we have already emphasized, this transformation is geometrically interpreted as a translation of a vertex of the tetrahedron patterned by the interaction. More precisely, the function \( V(x_i^f, x_j^f) \) imposes the variables \( x_i^f \), interpreted as edge vectors expressed in different frames, to match the metric of a Euclidean tetrahedron. The symmetry expresses the invariance of the matching condition under a translation of each of the vertices in an embedding of this tetrahedron in \( \mathbb{R}^3 \). This is also how the action of discrete residual of diffeomorphisms is encoded in 3d Regge calculus.

(ii) Group representation. The vertex function is given by the formula (4):

\[
V(g_i^f, g_j^f) = \int \prod_{\ell=1}^{4} dh_{\ell} \prod_{i=1}^{6} \delta((g_i^f)^{-1} h_{\ell} h_{\ell}^{-1} g_i^0).
\]

(32)
The invariance of this function under translations (27) of the vertex \( v_3 \) means that, for all \( \epsilon \in \mathfrak{su}(2) \),

\[
e_{G_{v_3}}(\epsilon) V(g_1^\ell, g_2^\ell) = V(g_1^\ell, g_2^\ell),
\]

(33)

where the argument of the plane wave is

\[
G_{v_3} = (g_1^\ell)^{-1} g_3(g_2^\ell)^{-1} g_4^\ell g_1^\ell.
\]

(34)

We thus see that translation invariance reflects, in the group representation, a conservation rule of \( G_{v_3} \) encode boundary holonomies, along paths connecting the center of each triangle \( \ell \) to its edges. In the interaction term, they define a discrete connection living on the graph dual to the boundary triangulation of the tetrahedron, which has the topology of a two-sphere. As illustrated on the right of Fig. 5, \( G_{v_3} \) is the holonomy along a loop circling the vertex \( v_3 \) of the tetrahedron. Thus, the symmetry under the translation of each vertex says the boundary connection is flat.

\[
\sum_{\{m'_\ell\}} \prod_{\ell} \delta_{n'_\ell, m'_\ell} V_{m'_\ell, n'_\ell} = \prod_{i} \delta_{n'_i, -n'_i} \sum_{\{j_1, j_2, j_3, j_4, j_5, j_6\}} \{j_1, j_2, j_3, j_4, j_5, j_6\}
\]

(37)

There is a connection, also pointed out in [12,13,41], between the flatness constraint described in (ii) and the topological identities (Biedenhard-Elliot) satisfied by the 6j symbol, which insures the formal invariance of the Ponzano-Regge spin foam model under refinement of the triangulation. To see how the symmetry relates to such identities, let us make, within the integral (35), the (trivial) substitution:

\[
V(g_1^\ell, g_2^\ell) \rightarrow \int dke_{G_{v_3}}(ke_3k^{-1}) V(g_1^\ell, g_2^\ell).
\]

(38)

\( G_3 \) is the vertex holonomy given by (34); the factor in front of \( V \) is the evaluation of a central function whose Plancherel decomposition is

\[
\int dke_{g}(ke_3k^{-1}) = \sum_{j} \chi^j(g) \tilde{\chi}^j(\epsilon_3).
\]

(39)

where \( \chi^j \) is the SO(3) character in the spin \( j \) representation, and \( \tilde{\chi}^j = \int dke^j(g)e_{\epsilon_3} \) is its group Fourier transform.\(^6\)

We also decompose into characters the three delta functions in the expression of \( V(g_1^\ell, g_2^\ell) \) [see Eq. (32)] associated to the edges \( i = 1, 3, 4 \) sharing the vertex \( v_3 \), with the Plancherel formula \( \delta(g) = \sum k d_k \chi^k(g) \). We thus obtain an expression in terms of the spins \( j_i^\ell \) and a sum over four additional spins \( k_i \) and \( j_i \). Elementary recoupling theory then shows that

\(^6\)Explicitly, \( \tilde{\chi}^j(\epsilon) = J_{2j+1}(|\epsilon|)/|\epsilon| \), where \( J_{2j} \) is the Bessel function of the first kind associated to the integer \( 2j+1 \), peaked on the value \( |\epsilon| = d_j \).
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\[
\sum_{\{ m \}} \prod_{\ell} t^\ell_{m_{\ell}} V^h_{m_{\ell}} n_{\ell} = \sum_{k_{i,j}} d_{k_1} d_{k_3} d_{j,\ell} \chi^j(\varepsilon_3)
= \sum_{k_{i,j}} d_{k_1} d_{k_3} d_{j,\ell} \chi^j(\varepsilon_3) \left\{ \begin{array}{c} 1, 2 \\ j, k_1 \\ j \\ j \\ k_2 \\ k_3 \\ 3 \end{array} \right\} \left\{ \begin{array}{c} 1, 2 \\ j, k_4 \\ j, k_3 \\ k_1 \end{array} \right\} \left\{ \begin{array}{c} k_1, k_3 \\ j, k_2 \\ j, j \\ j, j \end{array} \right\}.
\]

Comparing (36) and (40), we obtain the following identities:

\[
\forall e, \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \end{array} \right\} = \sum_{k_{i,j}} d_{k_1} d_{k_3} d_{j,\ell} \chi^j(e) \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j_5 \\ j_4 \\ j_6 \end{array} \right\} \left\{ \begin{array}{c} j_1 \\ j_5 \\ j_4 \\ k_1 \end{array} \right\} \left\{ \begin{array}{c} j_2 \\ j_3 \\ k_3 \\ k_4 \end{array} \right\}.
\]

In turn, these identities imply recursion relations for the same 6j symbols (see, e.g., [13,42]), interpreted as discrete versions of the Wheeler-DeWitt equation [12]. More generally, we expect that our type of analysis, based on GFT symmetries, can give a systematic way, also for gravity models in higher dimension, to derive algebraic identities of the spin foam quantum amplitude from the study of the GFT symmetries.

This gives a clear interpretation of the symmetry in the various representations of the GFT, which matches what we expect from the action of diffeomorphisms in discrete approaches. Thus, in the metric representation, the symmetry encodes the invariance under (noncommutative) translation of the four vertices of the tetrahedron. In the group picture, they encode the flatness of the discrete boundary connection, which is the Wheeler-DeWitt constraint in connection variables and thus the canonical diffeomorphism constraints. In the spin picture, they encode recursion relations for the fundamental spin foam amplitudes (6j symbols) and their behavior under coarse-graining.

E. GFT with vertex variables

We have seen that the invariance of the vertex function reflects some conservation rules for the holonomies \( G_v \) along loops surrounding the vertices of the tetrahedron patterned by the interaction. These conservation rules can be made manifest by integrating out the gauge group element \( h_i \) in the vertex function. Using three of the six delta functions in (4) to integrate three of the four integration variables \( h_i \), we obtain

\[
\begin{align*}
V(g^h, g^i) &= \int \prod_{i=1}^{4} dh_i \prod_{i=1}^{6} \delta((g^h_i)^{-1} h_i h_i^{-1} g^i_i) \\
&= \int [dh_4] \delta((g_4^h)^{-1} g_3^h (g_2^h)^{-1} g_6^h (g_5^h)^{-1} g_4^h) \delta((g_1^h)^{-1} g_3^h (g_2^h)^{-1} g_6^h (g_5^h)^{-1} g_4^h) \delta((g_5^h)^{-1} g_3^h (g_2^h)^{-1} g_6^h (g_5^h)^{-1} g_4^h) \\
&\quad \times \delta((g_5^h)^{-1} g_3^h (g_2^h)^{-1} g_6^h (g_5^h)^{-1} g_4^h).
\end{align*}
\]

Thanks to the normalization of the Haar measure, this simply gives

\[
V(g, g') = \delta(G_{v_i}) \delta(G_{v_2}) \delta(G_{v_3}).
\]

Note that the fourth constraint \( G_{v_4} = 1 \) is a consequence of the other three, due to the dependence relation \( G_{v_4}^{-1}(g_4 g_3^{-1})(G_{v_1} g_{v_4} G_{v_1})(g_3 g_4^{-1}) = 1 \) between the four vertex holonomies. This is the counterpart of the reducibility of the translation symmetry studied in the previous section. The dependence relation can be easily understood as a discrete Bianchi identity for the boundary connection on the boundary surface of the tetrahedron.

This form of the vertex function suggests yet another representation of GFT in terms of vertex variables \( v_i \in \mathfrak{so}(2) \) instead of edge vectors \( x_i \). These vertex variables, which are the generators of the translation symmetry, are introduced by plane-wave expansion \( \delta(G_{v_i}) = \int d^3 v_i \epsilon_{G_{v_i}}(v_i) \) of the delta functions on the group. Writing each of these plane waves as a cubic term, for, e.g.,

\[
\epsilon_{G_{v_i}} = \epsilon_{(g_i^h)^{-1} g_3^h} \epsilon_{(g_i^h)^{-1} g_6^h} \epsilon_{(g_i^h)^{-1} g_4^h},
\]

suggests to recast the GFT interaction in terms of new fields defined by the Fourier transform.
In terms of these new fields, well-defined on gauge invariant
path integral form of the Ponzano-Regge model. In

\[ \hat{\psi}_1(v_2, v_3, v_4) := \int dg_1 dg_2 dg_3 e_{g_1^{-1} g_2}(v_2) e_{g_2^{-1} g_3}(v_3) e_{g_3^{-1} g_4}(v_4) \varphi_1(g_1, g_2, g_3), \]

\[ \hat{\psi}_2(v_1, v_3, v_4) := \int dg_3 dg_4 dg_5 e_{g_3^{-1} g_4}(v_3) e_{g_4^{-1} g_5}(v_4) \varphi_2(g_3, g_4, g_5), \]

\[ \hat{\psi}_3(v_1, v_2, v_4) := \int dg_3 dg_4 dg_5 e_{g_3^{-1} g_4}(v_1) e_{g_4^{-1} g_5}(v_2) e_{g_5^{-1} g_4}(v_4) \varphi_3(g_3, g_2, g_5), \]

\[ \hat{\psi}_4(v_1, v_2, v_3) := \int dg_6 dg_7 dg_8 e_{g_6^{-1} g_8}(v_1) e_{g_7^{-1} g_8}(v_2) e_{g_8^{-1} g_8}(v_3) \varphi_4(g_6, g_4, g_1). \]

(44)

In this section, we investigate how the GFT symmetry described
above relates to the discrete residual action of diffeomorphisms in this model [11].

A. Diffeomorphisms in simplicial path integrals

The amplitude of a closed Feynman GFT diagram \( G \), in

\[ Z_\Delta = \int \prod_e dh \prod d^3 x e^{i S_\Delta(x, h)}, \]

(47)

where \( \Delta \) is the simplicial complex dual to \( G \), and \( S(x, h) \)

is the discrete 3d gravity action

\[ e^{i S_\Delta(x, h)} := e^{ \sum e \text{Tr}_H \, h_e = \prod e \text{Tr}_H (x_e)}. \]

(48)

The variables of this action are a discrete metric \( \{ x_e \}_{e \in \Delta} \)
on the edges of the triangulation and a discrete connection \( \{ h_e \}_{e \in \Delta} \) on the lines of \( G \). The group element \( H_e = \prod_{e \in \partial f} h_1 \) is the holonomy along the boundary of the

The action \( S_\Delta(x, h) \) is a discrete version of the continuum

\[ S(B, A) = \int \text{Tr} B \wedge F, \]

(49)

where \( B \) is the triad frame field and \( F \) is the curvature of the connection \( A \). We recall in the Appendix the local Poincaré symmetry of the continuum symmetry—namely, the SO(3) gauge invariance and translation symmetry:

\[ B \rightarrow B + d_A \phi \quad A \rightarrow A + d_A \times [A, X], \quad F \rightarrow F + [F, X], \]

(50)

with both \( X \) and \( \phi \) scalars with value in \( \mathfrak{su}(2) \). The translation
symmetry, typical of \( BF \)-type theories, is due to the Bianchi identity \( d_A F = 0 \). As we show in the Appendix, the action of diffeomorphisms in 3d gravity is classically equivalent to (a combination of gauge transformations and) translation of the frame field.

IV. FROM GFT TO SIMPLICIAL GRAVITY SYMMETRIES

We have seen in Sec. II that the amplitude of a Feynman
GFT diagram, in the metric representation, gives the
simplicial path integral form of the Ponzano-Regge model. In
The action $S_\Delta(x_e, h_l)$ enjoys a discrete version of these symmetries [11]. It can, moreover, be shown that, whenever $\Delta$ triangulates a three-manifold, the discrete residual of translation invariance, and hence of diffeomorphism invariance, in the discrete path integral (47), is partially\footnote{For a finer analysis of the divergences of the Ponzano-Regge model, see [26].} responsible for the large-spin divergences in the Ponzano-Regge model [11,32].

The discretization of the gauge transformations follows the usual lattice gauge theory techniques. The generators $A_v$ are labeled by vertices of the GFT graph $G$—equivalently by tetrahedra in the triangulation $\Delta$. The holonomy $h_l$ on the oriented lines of $G$ is transformed as $h_l \rightarrow \Lambda_v h_l \Lambda_v^{-1}$, where $v$, $l$ denote the source and target vertices of the line $l$. This means, in particular, that the holonomy $H_v$ around a boundary of a face $f_e$ is transformed as $H_v \rightarrow \Lambda_v H_v \Lambda_v^{-1}$, where $\Lambda_v$ is the generator associated to the reference vertex in $\partial f_e$, from which the holonomy is computed. The metric variable $x_e$ transforms as $x_e \rightarrow \Lambda_v x_e \Lambda_v^{-1}$. Such a gauge transformation, under which the action is clearly invariant, corresponds to a rotation of the reference frame of $e$.

The discrete residual of translation invariance is due to a discrete analogue of the Bianchi identity satisfied by the curvature elements $H_e$. In terms of the GFT diagram and its dual simplicial complex $\Delta$, this can be understood as follows. Given a vertex $v \in \Delta$, the set of GFT faces $f_e$ dual to the edges $e \supset v$ meeting at $v$ defines a cellular decomposition of a surface $L_v$, called the link of the vertex $v$. In GFT language, the link of a vertex is the boundary of a three-dimensional “bubble” of the diagram. Whenever the simplicial complex $\Delta$ defines a triangulated manifold (as opposed to a pseudomanifold), the link of every vertex has the topology of a two-sphere. Then, for any ordering of the edges $e \supset v$ meeting at $v$, the curvature elements $H_v$ satisfy a closure relation of the type

$$\prod_{e \supset v} (k^e_v)^{-1} H_e k^e_v = 1$$

(51)

for some group-valued functions $k^e_v := k^e_v(h_l)$ of the variables $h_l$ on the links $l$ of $L_v$. The group elements $k^e_v$ are interpreted as the parallel transport along paths between a fixed vertex in $L_v$ to the reference vertex of each face $f_e$ from which the holonomy $H_v$ computed. We have assumed here that the orientation of the faces $f_e$ agrees with a fixed orientation of the sphere $L_v$; if the orientation of $f_e$ is reversed, $H_v^{-1}$ should appear in place of $H_v$. Note that no such closure identity holds when the $L_v$ has a higher genus topology.

The idea of the works [11,32] was to use the identities (51) to prove a (commutative) discrete analogue of translation symmetry for the discrete action $S_\Delta = \sum e \text{Tr} x_e H_e$. To do so, the identity (51) is first written in terms of the projections $P_e := \text{Tr} H_e \bar{\tau}$ of the curvature elements onto the Lie algebra:

$$\sum_{e \supset v} (k^e_v)^{-1} (U^v_e P_e + [\Omega^v_e, P_e]) k^e_v = 0,$$

(52)

where the scalar $U^v_e$ and Lie algebra elements $\Omega^v_e$ are certain (complicated) functions of the $P_e$’s obtained from the Campbell-Hausdorff formula [11]. This leads to the invariance of $S_\Delta$ under the following transformation, generated by $\epsilon_v \in \mathfrak{su}(2)$:

$$x_e \rightarrow x_e + U^v_e \epsilon_v^e - [\Omega^v_e, \epsilon_v^e],$$

(53)

with $\epsilon_v^e(\epsilon_v^e) = k^e_v \epsilon_v^e(k^e_v)^{-1}$.

Note that, if $\epsilon_v^e$ is interpreted as a translation vector in the reference vertex frame of $L_v$, $\epsilon_v^e$ encodes the same translation vector parallel-transported in the reference frame of $f_e$. The transformation (53) is a discrete analogue of the translation symmetry (50).

In the next section, we show that the discrete Bianchi identity (51) can be related to a vertex translation symmetry in a direct way—that is, without any projection to the Lie algebra—provided one takes into account the noncommutativity of the translation algebra studied in Sec. III A. This will clarify the relationship between the GFT symmetry and the discrete Bianchi identities leading to diffeomorphism invariance in the simplicial path integrals.

B. Simplicial diffeomorphisms as quantum group symmetries

To see how the discrete Bianchi identities are tied to the invariance under noncommutative vertex translation defined in Sec. III C, let us fix the value $x_e$ of the metric in the exponential of the action (48) for all edges $e$ which do not touch the vertex $v$. This defines a function of the remaining $n_v$ variables in $\mathfrak{su}(2)$, labeled by the $n_v$ edges $e \supset v$ sharing $v$. Choosing an ordering of these edges, as in (51), one can lift this function to an element of the tensor product $\mathcal{C}(\text{SO}(3))^{\otimes n_v}$ of $n_v$ copies of $\mathcal{C}(\text{SO}(3))$.

Let us now act with the noncommutative translation

$$x_e \rightarrow x_e + \epsilon_v^e(\epsilon_v^e), \quad \epsilon_v^e(\epsilon_v^e) = k^e_v \epsilon_v^e(k^e_v)^{-1},$$

(54)

shifting the metric of the edges sharing $v$ by the variables $\epsilon_v^e$, defined as in (53). The group elements $k^e_v$ parallel transport the frame of a fixed vertex in $L_v$, and that of the reference vertex of the face $f_e$ from which the holonomy $H_v$ is computed. Upon such a translation, the function (48) gets transformed into a $\star$ product of functions of $\epsilon_v^e$:

$$\prod_e e^{\text{Tr} x_e H_e} \rightarrow \prod_{e \supset v} \star \prod_e e^{\text{Tr} (x_v + \epsilon_v^e) H_v(\epsilon_v^e)}. $$

(55)
Using the rule (10) for the * product of plane waves, we see that such a noncommutative translation acts on the action term by multiplication by the plane wave:

\[ e^{i \text{Tr}(e_\nu \prod_{e \in \partial} (k^e)^{-1} H, \xi^e))} = 1, \]  
(56)

which is trivial due to the Bianchi identity (51).

As we have seen in Sec. III C, both the GFT propagator and vertex functions, which the Feynman integrand (48) is built upon, are invariant under vertex translation \( x_e \to x_e + \epsilon_\nu \), for \( e \supseteq \nu \). The generator \( \epsilon_\nu \) is interpreted as a translation vector in a given frame. This is the frame associated to the reference point of the loop circling \( \nu \), along which the conserved holonomy is computed. In (30), for example, this is the frame associated to the edge 1 of the tetrahedron patterns by the interaction.\(^8\)

The transformation (55) has the same geometrical meaning: it corresponds to a vertex translation expressed in a given frame. This frame is the reference vertex frame of the link \( L_\nu \). Indeed, recall from the calculation of the Feynman amplitudes in Sec. II that the variable \( x_e \) present in the action term is the edge metric in the reference frame of the GFT faces \( f_e \) dual to the \( e \). Using the parallel transports \( k^e \nu \), one could instead use variables \( x^e_\nu \), labeling to the same edge metric but expressed in the reference vertex frame of \( L_\nu \). These are defined by \( x_e = k^e_\nu x^e_\nu (k^e_\nu)^{-1} \). Now, in this frame, a vertex translation acts as \( x^e_\nu \to x^e_\nu + \epsilon_\nu \), where \( \epsilon_\nu = (k^e_\nu)^{-1} \).

Thus, the equality (56), and hence the discrete Bianchi identity, express the invariance of the exponential of the action under the (quantum) GFT symmetry defined in the previous section. Note that, interestingly, the analysis of the invariance under simplicial diffeomorphisms distinguishes the closed GFT diagrams \( \mathcal{G} \) which define a manifold from those defining only a pseudomanifold. In fact, in the case of nonmanifold graphs, the triangulation has vertices \( \nu \) for which the link \( L_\nu \) defines a surface with nontrivial topology. For such vertices, there is no analogue of the discrete Bianchi identity (51): the invariance of the exponential of the action (48) under vertex translation is, therefore, broken.

The goal of the next subsection is to illustrate how these rather geometric considerations can be understood in a purely algebraic way. We will show on a simple example how the use of braiding techniques could give a systematic way to derive Bianchi identities from the GFT symmetry.

\(^8\)As made clear using the covariance of the plane wave upon conjugacy, the same translation expressed in a different frame, say, that of edge 3, is generated by \( \epsilon^3 = k e^3 g^{-1} \), where \( k := g^3 g_1 \) parallel-transport a one frame to another.

C. Noncommutative translations, invariance of the GFT amplitudes, and Bianchi identities

As spelled out in Sec. II, the integrand of a GFT Feynman amplitude in the metric representation is calculated by sticking together propagator and vertex functions along each loop \( f_e \) of the diagram, using the * product. This gives a product loop amplitude [see (18)]

\[ \prod_{f_e} \prod_{j=0}^{N+1} (\delta_{x^e_\nu} e_{b_{j+1}} (x^e_\nu)^{j+1}). \]  
(57)

The exponential of the discrete BF action (48) is then obtained by integrating, within each loop, over all metric variables \( x^e_\nu \), save one \( x_e := x^e_\nu \). It was shown in the previous section that the GFT propagator and vertex define invariant functions under the noncommutative translation (27). The question we are asking is to which extent the translation invariance of the propagator and vertex functions induces the invariance (55) and (56) of the action term. We will only sketch an answer here with a simple example, leaving the full proof to future work.

Since the transformation is quantum symmetry, it is crucial, to answer the above question, to keep track of the ordering of the variables in the calculation of the Feynman integrand. It is precisely to keep track of this ordering that the braided quantum field theory formalism [22] uses a perturbative expansion into braided Feynman diagrams.

Note that, to study the behavior of (57) under the translation of a vertex \( \nu \), it is enough to restrict the product to the set of loops \( f_e \) such that \( e \supseteq \nu \). This amounts to considering the contribution of a subdiagram called a “three-bubble” [23], which represents the vertex \( \nu \). A three-bubble, obtained by erasing all lines having strands of a given color, is a trivalent ribbon graph dual to the link \( L_\nu \) of a vertex of the dual triangulation.

Figure 6 shows the simplest GFT diagram of order two, dual to a triangulation of the sphere \( S^3 \) with two three-simplices; Fig. 7 shows the three-bubble obtained by erasing all the lines having strands of color 4. The three-bubble can be drawn as a braided diagram, on the left of Fig. 8: all vertices are put beside each other, all legs up, in the lower part of the diagram, in a way that preserves the cyclic order.
of the legs on the plane; then, the propagator strands, in the upper part of the diagram, connect the legs with each other. A convenient way to represent these vertices is as a product of three “cups” (see Fig. 8):

\[
\begin{array}{c}
\left( \begin{array}{ccc}
1 & 2 & 2 \\
3 & 3 & 1 \\
\end{array} \right)
\end{array}
\]  

(58)

The Feynman rules to compute the contribution of the three-bubble to the amplitude are easily read from (17). If one reabsorb the minus sign and group variables of the propagator into the vertex, we get a contribution of each “cup”

\[
\begin{array}{c}
\left( \begin{array}{c}
\begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array} \right)
\end{array}
\]

given by

\[
\mathcal{T}_\epsilon \triangleright \begin{array}{c}
\left( \begin{array}{ccc}
1 & 2 & 2 \\
3 & 3 & 1 \\
\end{array} \right)
\end{array} = e_{h_1^2 h_2^1 h_3 h_4^1 h_5^1 h_6^1} (\epsilon)
\]

(59)

whereas the propagator strands are just noncommutative delta functions \( \delta_\epsilon (x') \).

Upon noncommutative translation \( x \rightarrow x + \epsilon \), the “cup” function of two variables \( x_{ij} \) gets transformed as

\[
\mathcal{T}_\epsilon \triangleright \left( \begin{array}{c}
\begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array} \right) = e_{h_1^1 h_2^1} (\epsilon)
\]

(60)

This can be easily seen by group expansion of the noncommutative delta functions. One can then convince oneself that the translation invariance of the vertex function is then due to the following identities:

\[
\begin{array}{c}
\left( \begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array} \right) = e_{h_1^2 h_2^1 h_3 h_4^1 h_5^1 h_6^1} (\epsilon)
\]

(61)

Hence, we see that, by construction, the lower part of the braided bubble diagram defines a translation invariant function of the metric variables.

Now, the contribution of the three-bubble appears in the final integrand (57) as a product of loops, as drawn on the right of Fig. 8.

Going from the left to the right diagrams in Fig. 8 by “separating the loops” induces reordering of the strands—hence, a reordering of the metric variables. In order to probe the behavior of (57) under translation, the idea is to associate to a certain braiding map to the separation of the loops, induced by the universal R-matrix of \( DSO(3) \) given in (25).\footnote{Let us stress that this braiding is merely a technical aid to keep track of the action of our quantum group symmetry on functions of several Lie algebra variables and does not correspond here to any nontrivial braiding in the algebra of GFT fields, which we have chosen to be trivial, as in the standard GFT framework.} As a direct calculation shows, swapping two cups (the right one above the left one) with the \( DSO(3) \) braiding gives

\[
\begin{array}{c}
\left( \begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array} \right) = \left( \begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array} \right)
\]

(62)

where

\[
\begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array}
\]

denotes the action of \( h \) by conjugacy on the two variables of the cup:

\[
\begin{array}{c}
\left( \begin{array}{c}
\hphantom{1}
\hphantom{2}
\hphantom{2}
\hphantom{3}
\hphantom{1}
\end{array} \\
\begin{array}{c}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\hphantom{0}
\end{array}
\end{array} \right) = (\delta_{h_1^1 x_{ij}^1 h} \ast e_{h_1^1 h_2^1})(h_1^1 x_{ij})
\]

(63)
By construction, swapping the cups in this way intertwines the translation $T_{\epsilon}$.

Let us now use this braiding to “separate the loops.” The loop $12 \bar{1}$ is separated as follows:

$$e^{iTr[x_1H_{12}]}e^{iTr[h_{ij}^{-1}h_i^{-1}h_j^{-1}H_{j3}]}e^{iTr[h_{ij}^{-1}h_i^{-1}h_j^{-1}h_{kl}^{-1}]}.$$ \hspace{1cm} (65)

where $H_{ij} = h_{ij}^{-1}h_i^{-1}h_j^{-1}h_{ij}$ denotes the holonomy along the loop $ijj$. The invariance of this expression under the translation $x_{ij} \rightarrow x_{ij} + \epsilon$ expresses the invariance of the product

$$e^{iTr[x_1H_{12}]}e^{iTr[x_3^3H_{32}]}e^{iTr[x_1H_{13}]}$$

under the translation $x_{ij} \rightarrow x_{ij} + \epsilon_{ij}(\epsilon)$, where

$$\epsilon_{12}(\epsilon) := \epsilon, \hspace{1cm} \epsilon_{23}(\epsilon) := h_{2}^{-1}h_1e h_{1}^{-1}h_{2}^{-1}, \hspace{1cm} \epsilon_{31}(\epsilon) := h_{3}^{-1}h_1e h_{1}^{-1}h_{3}.$$ \hspace{1cm} (66)

In any case, the invariance leads to the identity (56), which reads here

$$e_{H_{12}h_{1}^{-1}h_jh_{1}^{-1}h_jh_{3}h_{3}^{-1}h_{1}(\epsilon) = 1. \hspace{1cm} (67)$$

This equality holding for all $\epsilon$, it gives a Bianchi identity of the type of (51):

$$H_{12}h_{1}^{-1}h_jh_{1}^{-1}h_jh_{3}h_{3}^{-1}h_{1} = 1. \hspace{1cm} (68)$$

We thus derived a Bianchi identity from the translation invariance of the vertex and propagator functions. In this analysis, the “twist” elements $k_{e}^j$ in the Bianchi identity, geometrically interpreted as parallel transport from a fixed point to the reference point of each loop, show up in commuting the variables using the $\text{DSO}(3)$ braiding.

The next step is to form the loop $23 \bar{2}$:

$$\text{Hence, using the $\text{DSO}(3)$ braiding to separate the loops finally gives a twisted product of loops:}$$

$$\text{(64)}$$

More generally, we expect an analogous algorithm for any planar three-bubble; namely, when the link $L_{ij}$ of the vertex has the topology of a two-sphere. Starting from the three-bubble drawn as a braided diagram, the bottom part (a product of “cups”) gives, by construction, a translation invariant function of the metric variables $x_{ij}$. The algorithm will then define a sequence of topological moves corresponding to the separation of the loops and inducing a reordering of the variables $x_{ij}$, and associate to it a certain $\text{DSO}(3)$ braiding map. This braiding map encodes the behavior of the amplitude (57) under noncommutative translation. The example shown above is particularly simple, as the braided three-bubble does not involve any crossing of the propagator strands; in general, an additional rule should be added in the definition of the braiding map, which would take into account such crossings.

Just as in the above example, the action of such a braiding map on the function defined by the product of cups will induce a twisting of the variables $x_{ij}$ by certain group-valued functions $k_{e}^j(h)$ of the holonomies. A condition to obtain Bianchi identities, and hence an invariance of the action term $e^{iS_{3}(x_{ij})}$, is that these functions do not depend on the variables $j$ within a loop $e$: namely, $k_{e}^j := k_{e}$. We conjecture that this condition can be reached precisely when the three-bubble is planar—namely, when all the crossings of the braided diagram are removable by some topological move. In the presence of nontrivial crossings, on the other hand, the braiding map will give an invariance of the amplitude (57), which will not translate into any Bianchi identity or an invariance of the action term $e^{iS_{3}(x_{ij})}$ [obtained from (57) by integration over all the variables $x_{ij}$, save one per loop]. This reflects a breaking of the discrete diffeomorphism symmetry whenever the (closed) GFT graph has nonspherical three-bubbles—namely, for pseudomanifold graphs.

Whether this conjecture can be proven remains to be seen: we leave this for future work. It will also be important
to understand how this analysis is affected by the use of a nontrivial braiding in the algebra of GFT fields, intertwining the quantum symmetry.

D. Open diagrams and $n$-point functions

The geometrical and algebraic analysis of the previous two sections can be extended to open GFT graphs, with fixed boundary metric or connection data. An open GFT graph is dual to a simplicial complex with boundaries. We have seen that the invariance of the Feynman integrand under noncommutative translation of a vertex $\nu$ of this simplicial complex is due to a discrete Bianchi identity on the link $L_{\nu}$ of the vertex. We showed both geometrically and algebraically that, when $\nu$ is in the bulk, the invariance holds only when $L_{\nu}$ has a trivial topology, or, equivalently, when the three-bubble associated to $\nu$ is planar.

The same condition applies when the vertex lies at the boundary. In this case, the link $L_{\nu}$ defines an open surface, whose boundary is a loop circling the vertex $\nu$: this is the boundary of the graph triangulation. Now, in the “group” representation, the boundary data encodes a boundary condition. One can then easily convince oneself that a discrete “Bianchi identity” on the link $L_{\nu}$ simply says that the holonomy of this boundary connection, along $\partial L_{\nu}$, is trivial. Such a discrete Bianchi identity, and hence the invariance of the Feynman integrand under noncommutative translation of the vertex, hold when the link $L_{\nu}$ has a trivial (disk) topology.

We had already noticed, at the level of the GFT vertex, that our symmetry implies (in the group representation) flatness of the boundary connection. In fact, dealing with a flat boundary connection means that the holonomies along all (3d) contractible loops are trivial. Now, the loop $\partial L_{\nu}$ circling the boundary vertex $\nu$ is contractible precisely when the link $L_{\nu}$ has a trivial topology: the invariance under translation then holds and expresses precisely that the holonomy is trivial. Thus, the behavior of the Feynman integrand under noncommutative translation of the boundary vertices indeed encodes the flatness of the boundary connection—namely, what we expect as a result of diffeomorphism invariance.

More generally, for the GFT graphs dual to manifolds, the behavior of the Feynman amplitudes under our quantum GFT symmetry is consistent with what we know about discrete diffeomorphisms at the quantum level from canonical (discrete) 3d gravity, as well as its covariant path integral formulation. Since not much is known about the action of diffeomorphisms in simplicial gravity on pseudomanifold, we conclude that we are not missing any expected feature of discrete diffeomorphism invariance, in our trivially braided GFT formalism, as far as it can be seen at the present stage of development.

Given the interpretation of our GFT symmetry as the counterpart of diffeomorphism invariance, it is natural to ask whether the GFT $n$-point functions respect the symmetry. We know this is not the case: sticking to the usual GFT formalism, we have used a trivial braiding in the algebra of fields, which does not commute with the action of our symmetry transformations. As it is well-known, this leads generically to a breakdown of the symmetry at the quantum level. In the context and spirit of the braided quantum field theory formalism, it would be more natural to use a nontrivial braiding intertwining the symmetry and hence fully implement the covariance of the $n$-point functions. However, the consequences of such a nontrivial braiding—although currently under investigation—are difficult to forecast, at this stage. In fact, it should clear from the above analysis of the amplitudes that the properties of GFT $n$-point functions in this trivially braided GFT context do not seem to indicate inconsistencies, a specific physical reason why a nontrivial braiding would be necessary, or any problem with the implementation of diffeomorphism invariance. On the contrary, none of the expected features of diffeomorphisms seems to be missing in this formalism.

V. DIFFEOMORPHISMS IN TOPOLOGICAL MODELS IN HIGHER DIMENSIONS

The analysis of the previous sections can be extended to higher dimensions, for models describing BF theory, in a rather straightforward manner. Here, we consider the Ooguri GFT [41] for 4d BF theory, generalized to include colors. The variables are complex scalar fields $\varphi_\ell$, with $\ell = 1, \cdots, 5$, defined on $G^{\otimes 4} = SO(4)^{\otimes 4}$, which satisfy the gauge invariance condition

$$\forall h \in SO(4), \quad \varphi_\ell(h g_1, h g_2, h g_3, h g_4) = \varphi_\ell(g_1, g_2, g_3, g_4) \quad \forall \ell.$$ (69)

The action of the model is $S[\varphi] = S_{\text{kin}}[\varphi] + S_{\text{int}}[\varphi]$, with

$$S_{\text{kin}}[\varphi] = \int [dg]^4 \sum_{\ell=1}^{4} \varphi_\ell(g_1, g_2, g_3, g_4) \varphi_\ell(g_1, g_2, g_3, g_4).$$

$$S_{\text{int}}[\varphi] = \lambda \int [dg]^10 \varphi_1(g_1, g_2, g_3, g_4) \varphi_2(g_4, g_5, g_6, g_7) \varphi_3(g_7, g_3, g_8, g_9) \varphi_4(g_9, g_6, g_2, g_{10}) \varphi_5(g_{10}, g_8, g_5, g_1)$$

$$+ \lambda \int [dg]^10 \bar{\varphi}_5(g_1, g_5, g_8, g_{10}) \bar{\varphi}_4(g_{10}, g_2, g_6, g_9) \bar{\varphi}_3(g_9, g_8, g_3, g_7) \bar{\varphi}_2(g_7, g_6, g_5, g_4) \bar{\varphi}_1(g_4, g_3, g_2, g_1).$$ (70)
the decompositions $g$ into simplicial interpretations. The field $\varphi_t(g_1, \ldots, g_4)$ represents a three-simplex (tetrahedron), its four arguments being associated to its boundary triangles. The interaction encodes the combinatorics of five such tetrahedra glued pairwise along common triangles to form a four-simplex. The kinetic term encodes the gluing of four-simplices along shared three-simplices.

The group Fourier transform giving the metric representation is easily extended [15] to functions of (several copies of) $SO(4) \sim SU(2) \times SU(2)/\mathbb{Z}_2$,

$$\hat{\varphi}_t(x_1, \ldots, x_4) = \int [dg]^4 \varphi_t(g_1, \ldots, g_4)e_{g_1}(x_1)\ldots e_{g_4}(x_4),$$

$x_i \in \mathfrak{so}(4) \sim \mathbb{R}^6$. (71)

The plane waves $e_g \mapsto \mathfrak{so}(4) \sim \mathbb{R}^6 \mapsto U(1)$ are defined as the product of $SU(2)$ plane waves, defined in Sec. II, using the decompositions $g = (g_-, g_+)$ and $x = (x_-, x_+)$ of the group and $\mathfrak{so}(4)$ algebra elements into left and right components:

$$e_g(x) = e^{iTr_x g}e^{-iTr_x g^\dagger}. \quad (72)$$

The $\star$ product is the Fourier dual of the convolution product of $SU(2)$ introduced in Sec. II. The variables $x$ are geometrically interpreted as bivectors that the standard lattice $BF$ theory assigns to triangles, in each tetrahedron. Just as in $3d$, the gauge invariance condition (69) is dual, upon a Fourier transform, to a closure constraint $\hat{C}(x_1, \ldots, x_4) = \delta(\sum_{i=1}^4 x_i)$ of the four field variables, imposed by a noncommutative delta function defined as in (13).

By extending the $3d$ symmetry analysis to the $4d$ case, we will consider the action of rotations and translations of the quantum double $^{10}$ $D$$SO(4)$ on the scalar fields $\varphi_t$. The action of the double on fields over the group is the same we presented in Sec. III A. Thus, an element $f \otimes \Lambda$, with $f \in \mathcal{C}(SO(4))$ and $\Lambda \in SO(4)$, acts on a function $\hat{\varphi} \in \mathcal{C}(SO(4))$ as

$$\phi(g) \mapsto \phi(\Lambda^{-1}g\Lambda), \quad \hat{\phi}(g) \mapsto f(g)\hat{\phi}(g) \quad (73)$$

and dually on its group Fourier transform $\hat{\phi}(x)$ by conjugacy and translation of the Lie algebra variable $x$.

As in the Boulatov case, we easily check that the only gauge covariant action of rotations which leaves the interaction term invariant is the diagonal rotation: In the metric formulation, gauge covariance simply means that a rotation preserves the closure $\delta(\sum_{i=1}^4 x_i)$ of the bivectors.

The realization of the translation symmetries is analogous to $3d$, except that now they act at the edges of the simplices patterned by the fields, rather than the vertices. The transformations are thus generated by four $\mathfrak{so}(4)$ translation parameters $e_{\ell r}$, where $e$ labels the ten edges of the interaction four-simplex, diagrammatically represented by its dual diagram in Fig. 9. Each edge of this four-simplex is represented by an subdiagram called an “edge graph”. Thus, if $e_{\ell r}$ denotes the unique edge that does not belong to the tetrahedra $\ell$ or $\ell'$, the edge graph associated to $e_{\ell r}$ is obtained by removing all the lines which contain strands of color $\ell$ or $\ell'$. The edge graph of $e_{34}$ is pictured in Fig. 10: its three lines represent the three triangles 1, 3, 4 sharing $e_{34}$.

To define the action of a translation of the edge $e_{34}$, we equip the lines of the edge graph with an orientation, as drawn in Fig. 10. Using this convention, each line has an “incoming” and an “outgoing” external strand. A translation of $e_{34}$, generated by $e_{34} \in \mathfrak{so}(4)$, acts nontrivially only in the strands of the edge graph. In the metric representation, it shifts the corresponding variables $x_i^\ell$ by $\pm e_{34}$, whether the strand $i$ comes in or out of $e_{34}$:

$$x_i \mapsto x_i - e_{34} \quad \text{if } i \text{ is outgoing}, \quad x_i \mapsto x_i + e_{34} \quad \text{if } i \text{ is incoming}. \quad (74)$$

---

$^{10}$Note that a priori we could choose a bigger quantum group, like a deformation of the Poincaré group in six dimensions. The classification of quantum symmetries for noncommutative spaces has been only partially completed in $4d$ [44]. Deformations of symmetries for higher-dimensional spaces have still to be explored. In our case, the choice of the quantum group of interest is dictated by the kinematical phase space of $4d$ $BF$ theory and by its known discrete classical symmetries, which we want to encode at the GFT level.
in a way that preserves the closure $\delta(\sum_{i=1}^{10} x_i^j)$ of each tetrahedron. More precisely, the translation $T_{e_{34}}$ of the edge $e_{34}$ acts on the dual fields as

\[
\begin{align*}
T_{e_{34}} \phi_1(x_1, x_2, x_3, x_4) &= \star_{e_{34}} \phi_1(x_1 - e_{34}, x_2, x_3, x_4 + e_{34}), \\
T_{e_{34}} \phi_2(x_4, x_5, x_6, x_7) &= \star_{e_{34}} \phi_2(x_4 - e_{34}, x_5 + e_{34}, x_6, x_7), \\
T_{e_{34}} \phi_3(x_{10}, x_8, x_5, x_1) &= \star_{e_{34}} \phi_3(x_{10} - x_8, x_5 - e_{34}, x_1 + e_{34}), \\
T_{e_{34}} \phi_\ell &= \phi_\ell \quad \text{if} \quad \ell = 3, 4.
\end{align*}
\]

(75)

The same field transformation is expressed in a more explicit way (without star product) in the group representation, as follows:

\[
\begin{align*}
T_{e_{34}} \varphi_1(g_1, g_2, g_3, g_4) &= e_{e_{34}} (e_{34}) \varphi_1(g_1, g_2, g_3, g_4), \\
T_{e_{34}} \varphi_2(g_4, g_5, g_6, g_7) &= e_{e_{34}} (e_{34}) \varphi_2(g_4, g_5, g_6, g_7), \\
T_{e_{34}} \varphi_3(g_{10}, g_8, g_5, g_1) &= e_{e_{34}} (e_{34}) \varphi_3(g_{10} - g_8, g_5 - e_{34}, g_1 + e_{34}), \\
T_{e_{34}} \varphi_\ell &= \varphi_\ell \quad \text{if} \quad \ell = 3, 4.
\end{align*}
\]

(76)

We see that this transformation matches the intuition corresponding to translating bivectors $[so(4)$ Lie algebra elements] associated to the triangles of the four-simplex dual to the GFT interaction vertex, by means of Lie algebra valued generators associated to its edges. This matches also the action of diffeomorphisms on the bivectors of discrete BF theory [recall that the transformations we have defined take the closure condition (metric compatibility) into account].\footnote{Unlike the 3d case, however, we have no geometric description in terms of translating the edges of a four-simplex embedded in four-dimensional flat space. This is only to be expected, given that we are dealing with a nongeometric theory and thus with nongeometric four-simplices.} It can be checked by direct calculation that the GFT action (70) is invariant under the above field transformations.

In fact, one verifies, as in the 3d case, that both kinetic and vertex functions themselves are left invariant—before integration. A way to make this invariance manifest is to extract from, say, the vertex function in group variables the conservation laws for the holonomies associated to edges of the four-simplex dual to the GFT vertex. Just as in Sec. III E, the explicit integration over the group elements $h_\ell$ in the vertex gives

\[
\begin{align*}
V(g, g') &= \int \prod_{i=1}^{5} (dh_i) \prod_{j=1}^{10} \delta(g_j^i h_i h_i^{-1} (g_j'^i)^{-1}) \\
&= \delta(G_{12}) \delta(G_{13}) \delta(G_{15}) \delta(G_{23}) \delta(G_{25}) \delta(G_{35}),
\end{align*}
\]

(77)

We recognize here the $G_{ij}$ as the holonomies around the edges $(ij)$. The delta functions in (77) encode the flatness conditions which, as expected from the canonical analysis of discrete BF theory, constrain the connection variables as a result of the diffeomorphism symmetry.

Note that the holonomies associated to the edges $(ii)$ are missing. This is analogous to the 3d case where the translations of the four vertices of the tetrahedron are not all independent; only three of them are. Also, in the 4d case, the translations of the edges are not all independent, just as the continuum symmetry can be shown to be reducible (cf. the Appendix): this is due to the Bianchi identities satisfied by the boundary connection represented by the field variables. In fact, one can prove that translating a vertex, i.e., translating all edges sharing this vertex, leaves invariant the interaction term and, by extension, the integrand of the Feynman amplitude. The true symmetry is, therefore, represented by the above edge translations modulo the translations of the edges following a vertex translation.

We thus see that, for (the GFT model describing) 4d BF theory, everything proceeds in parallel with the 3d case, the only new ingredient being the reducibility of the resulting symmetry. However, the strategy used here to define the action of diffeomorphisms in GFT can, in principle, be extended to the physically more interesting case of 4d gravity GFT models, obtained by constraining the topological one [5,6]. In general, we expect that the imposition of the simplicity constraints will break the full symmetries of Ooguri’s GFT. It will be interesting to determine whether there is an eventual remnant symmetry and, if not, whether the vertex translations become then the relevant, if only approximate, symmetry [29]. In this case, such a symmetry could admit a good geometric interpretation as translations of the vertices of a geometric four-simplex in an embedding 4d flat space, as we expect from diffeomorphisms in discrete gravity [7,10].
VI. ADDITIONAL INSIGHTS

We now discuss additional insights that the newly identified GFT symmetry provides, concerning various aspects of the GFT formalism itself. While these are somewhat secondary results, we believe they confirm the importance of the new symmetry and suggest that further progress can be triggered by its identification.

The necessity of coloring. The introduction of coloring in GFT models in [23] has already been proven useful in studies of the topological properties of the Feynman diagrams generated by such models [24,25,45]—in particular, for the automatic removal of complexes with some types of extended singularities that are instead generated by the noncolored models. Most important, it has been crucial for the proof that the 3d GFT we have studied admits a topological expansion of its Feynman diagrams such that manifold configurations of trivial topology dominate the sum for large values of the representation cutoff [28]. These important results have important implications for the program of GFT renormalization and for defining a GFT generalization of the notion of the (double) scaling limit of matrix models [39], and thus for the understanding of the continuum limit. No obvious physical or geometric relevance, however, had been discovered, until now, for the same coloring. Our results show, on the other hand, that coloring is a necessary feature of GFT models for 3d gravity and general BF theories. In fact, coloring is a necessary ingredient in the definition of the GFT diffeomorphism symmetry we have identified and discussed in this paper. More precisely, it can be shown that removing the coloring leads to the immediate breaking of the symmetry and that only a restricted translation of the vertices of the tetrahedron dual to the GFT vertex can be defined as a field transformation leaving the noncolored action invariant, such as the one identified in [21]. This symmetry, however, although being a particular combination of the symmetry transformation we have studied, does not have a clear simplicial gravity interpretation. Given the interpretation of our GFT symmetry as the counterpart of discrete diffeomorphisms in simplicial gravity path integrals, the importance of coloring from the physical/geometrical point of view becomes instead manifest. In its light, we recognize the colored Boulatov GFT model as the correct GFT description of 3d quantum gravity.

A braided group field theory formalism? In this paper, we have studied the issue of diffeomorphism symmetry within the standard (colored) group field theory formalism. In particular, the algebra of fields we have worked with has been assumed to have trivial braiding [20,22]; i.e., the map between the tensor product of two fields and the one with opposite ordering is given by the trivial flip map. At the same time, however, the symmetry we have identified in the GFT action corresponds, as we have stressed, to a quantum group symmetry acting on this space of fields. As such, its action on the space of fields would naturally induce, when these are defined as elements in its representation category, a nontrivial braiding structure [46]. This also results in a corresponding braided statistics [47]. Most important, it can be shown that the use of the induced braiding map in the algebra of fields is necessary, if the symmetry is to be preserved at the quantum level [36,37,46], for example, so that the correct Ward identities for n-point functions follow from the existence of the symmetry at the level of the action. We will discuss briefly below whether this is necessary on physical grounds in our context and what the properties of the n-point functions are in our trivially braided context. In any case, the above considerations suggest, at least from a mathematical and field theoretic perspective, to consider a generalization of the GFT formalism, beyond the one as noncommutative field theories, achieved in [15], to a braided noncommutative group field theory (see also [21] for further arguments in this direction). The first issues to tackle, in this direction, are 1) what is the correct braiding among GFT fields intertwining our quantum group symmetry, if it exists at all; and 2) what are the physical consequences of the implementation of a nontrivial braiding and of the resulting quantum Ward identities, from the point of view of simplicial quantum gravity, loop quantum gravity, and spin foam models.

Constraints on GFT model building. Another useful role that symmetries play in usual quantum field theories is that they help constraining the allowed field interactions. In fact, the requirement that the GFT interactions preserve the quantum group symmetry we identified as discrete diffeomorphisms rules out some GFT interactions that could be considered, a priori, as admissible.

We have already discussed above how removing the coloring form the GFT fields, i.e., considering the original Boulatov formulation with a single field, breaks the symmetry. This can also be understood as a special case of a larger set of possible GFT interactions within the colored GFT formulation, that we now see to be ruled out by symmetry considerations. The colored model we worked with, for 3d gravity, was based on four different fields \( \varphi_i \), with \( l = 1, 2, 3, 4 \), and the only interaction term was of the form \( \varphi_1 \varphi_2 \varphi_3 \varphi_4 \) (plus complex conjugate), with standard tetrahedral combinatorics of arguments. The single-color Boulatov interaction corresponds to terms of the type \( \varphi_1 \varphi_2 \varphi_3 \varphi_4 \). A quick calculation shows that, not only such terms, but any interaction being more than linear in any of the colored fields (e.g., \( \varphi_1 \varphi_2 \varphi_3 \varphi_4 \) or \( \varphi_1 \varphi_2 \varphi_3 \varphi_4 \)) is not invariant under our GFT diffeomorphism symmetry. We are then left with interactions that involve linearly all the \( d \) GFT fields (in models generating \( d \)-dimensional simplicial structures). The ordering of such fields can be chosen at will (in our trivially braided context).

We can, however, also ask whether the ordering of the (group or noncommutative) arguments of such fields in the interaction term can be chosen at will. Different orderings,
in fact, have been considered in the literature (see [39]). In colored models, the order of the arguments of the field is considered as fixed and does not play any special role (the orientability of the resulting Feynman diagrams is already ensured by the complex structure and by the requirements of same-color propagation only [45,48]). Regarding the interplay between the ordering of arguments and symmetry, the situation is slightly trickier. It can be seen easily that, for any given choice of ordering, there exists a (set of) transformation(s) acting on the $d + 1$ fields leaving the action invariant and corresponding to diffeomorphisms, in the sense we have discussed. The very definition of the transformations retains the imprint of the chosen ordering of field arguments. At the same time, however, it can be shown that such a transformation would not, in general, leave invariant a vertex defined by a different ordering nor an action involving a sum over different orderings. This means, for example, that the GFT field itself cannot be defined to be invariant under permutations of its arguments, as this imposition would break its covariance under the diffeomorphism transformation and then the invariance of the action. It must be said, however, that a possible way out of this restriction could be, once more, an appropriate braiding that relates fields defined with different orderings of their arguments and, possibly, intertwines our symmetry. We leave this for future work.

Last, one could consider defining both higher-order interaction terms, i.e., terms of order higher than $d + 1$ involving colored fields, with various choices of pairing of field arguments, as well as other terms still of order $d + 1$, but defined by nontetrahedral combinatorics of arguments. Our symmetry severely constrains model-building of this type. We have not performed yet a complete analysis. However, we have considered some examples. One interesting example of an alternative interaction term, the so-called “pillow” term, has been introduced in [49] and studied further in [43]. It has the following form (in its colored version):

$$\frac{\lambda}{4^d} \prod_{i=1}^{6} \int dg \phi_1(g_1, g_2, g_3) \phi_2(g_3, g_4, g_5) \times \phi_3(g_4, g_2, g_6) \phi_4(g_6, g_5, g_1).$$  \hspace{1cm} (78)

So, it is given by the same type of vertex function, i.e., a product of delta functions on the group, enforcing the identification of edge variables among four triangles, as in the standard tetrahedral term. However, the combinatorial pattern is now different and corresponds to two pairs of triangles glued to one another along two edges in each pair and along one single edge between the two pairs. The interest in the addition of such a term lies in the fact that it turns the (noncolored) Boulatov model into a Borel summable one [with some restrictions on the coupling constant $\delta$ and with a different (worse) scaling behavior]. It can be proven, however, that this term is not invariant under GFT diffeos and thus is not an admissible modification of the action of the model, in the colored case.

We stress again that the above considerations would be modified by the introduction of a nontrivial braiding among fields, with a corresponding generalization of the GFT formalism. However, not knowing the correct braiding structure, it is impossible to be more definite about what the modifications would be.

**VII. CONCLUSIONS**

Using the recently introduced noncommutative metric formulation of group field theories, we have identified a set of GFT field transformations, forming a global quantum group symmetry of the GFT action and corresponding to translations of the vertices of the simplices dual to the GFT interaction vertex, in a flat space embedding. The analysis of the action of these transformations at the level of the GFT Feynman amplitudes, which are given, in this metric formulation, by simplicial gravity path integrals, shows that the transformations we identified correspond to (the discrete analogue of) diffeomorphisms for fixed simplicial complex satisfying manifold conditions and leave the same amplitudes invariant thanks to discrete Bianchi identities, whose GFT origin we are now able to exhibit. Moreover, for open Feynman diagrams dual to simplicial manifolds with boundaries, we have shown that the same transformations enforce the flatness of the boundary connection and thus encode the simplicial version of the canonical gravity constraints, as expected.

While we focus on the case of 3d Riemannian gravity, we also show how our results generalize straightforwardly to $BF$ theories in higher dimensions. Thus, our results, on the one hand, match those obtained, concerning discrete diffeomorphisms, in the context of simplicial gravity (e.g., Regge calculus) and, on the other hand, improve them by both embedding them within a more general context and rephrasing them in purely (quantum) field theoretic language. An immediate advantage of this embedding is the clear way in which we can now link to one or another various aspects of diffeomorphism invariance in spin foam models, canonical loop quantum gravity, and simplicial gravity, previously discussed in the literature and now understood to be all consequences and manifestations of the same GFT field symmetry: the symmetry of the Regge action and the simplicial Bianchi identities (manifest in the metric representation of GFTs), the canonical constraints of loop quantum gravity (adapted to a simplicial complex, best seen in the group picture), and the algebraic identities satisfied by $nj$ symbols and at the root of the topological invariance of state sum (spin foam) models (obtained from the GFT symmetry in representation space).

Our analysis also provides some new insights on the GFT formalism itself. These include the need for coloring in the GFT formalism, from the point of view of simplicial gravity symmetries; the possible role of braiding in this
ACKNOWLEDGMENTS

We thank V. Bonzom, S. Carrozza, B. Dittrich, E. Livine, R. Oeckl, and A. Perez for useful comments and discussions. D. O. gratefully acknowledges financial support from the A. von Humboldt Stiftung.

APPENDIX: \(BF\) ACTION AND ITS SYMMETRIES

In this appendix, we recall the standard basic facts about the symmetries associated to the \(BF\) action.

We work with a \(d\)-dimensional manifold \(\mathcal{M}\), equipped with a principal bundle associated with the semisimple Lie group \(G\). The Lie algebra of \(G\) is noted as \(\mathfrak{g}\) and is equipped with a nondegenerate Killing form \(\kappa\) which we note as \(\kappa\). The curvature two-form of the connection \(A\) is a scalar field with value in \(\mathfrak{g}\). The \(BF\) action is built using the Killing form \(\kappa\), which is a scalar field with value in \(\mathfrak{g}\) and is equipped with a nondegenerate Killing form which we note as \(\kappa\).

The \(BF\) action is invariant under the adjoint action of the diffeomorphisms as

\[ A \rightarrow A + \delta_Q A = A, \]
\[ B \rightarrow B + \delta_Q B = B + d_A \Phi = B + d\Phi + A \wedge \Phi. \]

\[ \Phi \rightarrow \Phi + \Phi' \] with value in \(\mathfrak{g}\), generates on shell the same transformation, since \(d_A \Phi = d_A \Phi'\), due to \(d_A^2 V = [F, V]\), which last term is zero on shell.

The \(BF\) action is clearly invariant under the diffeomorphisms, since it is purely topological. Let us consider explicitly the (infinitesimal) action of the diffeomorphisms. Considering a vector field \(\xi\), the infinitesimal action of the diffeomorphisms is given by the Lie derivative \(L_\xi\). We have, therefore,

\[ B \rightarrow L_\xi B = d(\iota_\xi B) + \iota_\xi (dB), \]
\[ A \rightarrow L_\xi A = d(\iota_\xi A) + \iota_\xi (dA), \]

where we have introduced the interior product \(\iota_\xi\), which satisfies, in particular,

\[ \iota_\xi (\omega_1 \wedge \omega_2) = \iota_\xi (\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge \iota_\xi (\omega_2), \]

with \(\omega_1\) and \(\omega_2\), respectively, a \(p\) and a \(q\) form. These transformations can actually be related to the previous transformations (A3) and (A4). We have that

\[ \iota_\xi (d_A B) = \iota_\xi (dB) + \iota_\xi (A \wedge B) = \iota_\xi (dB) + [\iota_\xi (A), B] - A \wedge \iota_\xi (B), \]
\[ \iota_\xi (F) = \iota_\xi (dA) + \iota_\xi (A \wedge A) = \iota_\xi (dA) + [\iota_\xi (A), A]. \]

Taking \(X = \iota_\xi B\) and \(\Phi = \iota_\xi A\), we can reexpress the action of the diffeomorphisms as

\[ L_\xi B = \delta^L_{\iota_\xi A} B + \delta^T_{\iota_\xi B} B + \iota_\xi (d_A B), \]
\[ L_\xi A = \delta^L_{\iota_\xi A} A + \delta^T_{\iota_\xi B} A + \iota_\xi (F). \]

This means that on shell (A2), the diffeomorphism action is equivalent to the translation (A4) and gauge transformation (A3).