Fluctuation Spectrum for Linear Gyroviscous MHD

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Abstract:

The influence of gyroviscosity on the fluctuations of an MHD plasma is investigated by the method reported in a previous note \(^1\). The main result is that gyroviscosity does not help to remove ultraviolet divergences. For a sub-class of observables it does not even show up. The full non-linear problem may be needed.
A general formalism for obtaining the fluctuation spectrum of plasmas and fluids in statistical equilibrium has been proposed in Ref. 1. It is valid for linearized equations of conservative systems in a heat bath which allows the use of Gibbs statistics. The application of this formalism to gyroviscous one-fluid homogeneous plasmas is discussed in this letter. For general observables the spectrum depends on the eigenvalues of a symmetric operator containing gyroviscous and ideal MHD contributions. For a more restricted class of fluctuations such as density fluctuations, however, the spectrum only depends on the eigenvalues of the MHD operator.

The linearized equations of motion of non-dissipative gyroviscous one-fluid plasmas can be written $^2$ in the form

$$\rho \ddot{\mathbf{v}} + \nabla \cdot \mathbf{S} + \mathbf{Q} \dot{\mathbf{v}} = \mathbf{0}$$

(1)

where $\mathbf{v}$ is the perturbed fluid velocity, $\rho$ is the mass density in equilibrium, $\mathbf{Q}$ the MHD operator $^3$ and $\mathbf{S}$ the perturbed gyroviscous tensor $^4$.

It is assumed that the unperturbed plasma is in homogeneous static equilibrium with a constant magnetic field $\mathbf{B}_0$, and that the perturbations are two-dimensional with velocities perpendicular to $\mathbf{B}_0$. For

$$\mathbf{B} = \mathbf{B}_0 \mathbf{e}_\phi$$

(2)

the components of the tensor $\mathbf{S}$ are $^4$.
\[ s \nabla^2 \pi = -s \pi_y \gamma = -\alpha \Gamma_x y, \]  
(3)

\[ s \nabla \pi_y = s \nabla \gamma x = \frac{-s}{2} \left( \Gamma_y - \Gamma_x x \right), \]  
(4)

with

\[ \Gamma_x y = \frac{1}{2} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right), \]

\[ \Gamma_y y - \Gamma_x x = \frac{\partial \psi_y}{\partial y} - \frac{\partial \psi_x}{\partial x}, \]

\[ \alpha = \frac{P_0}{\omega_{ie}}, \]  
(5)

\[ P_0 \] and \( \omega_{ie} \) being the pressure and the ion cyclotron frequency in equilibrium, respectively.

It is a matter of simple algebra to write

\[ \nabla \cdot \pi \pi = \frac{\alpha}{2} \nabla^2 \left( \frac{\pi}{\psi} \times \nabla \psi \right) \]  
(6)

and

\[ \nabla \cdot \pi V = -\left( \frac{\psi}{\mu_0} + \gamma P_0 \right) \nabla \left( \psi \cdot \nabla \psi \right) \]  
(7)

where \( \gamma \) is the ratio of specific heat capacities.

Equation (1) can now be written as

\[ \ddot{V} + E \dot{V} + F V = 0, \]  
(8)

where

\[ E = \alpha \nabla^2 \frac{\pi}{\psi} \times \]  
(9)
is an antisymmetric operator and

$$ F = -b \nabla ( \nabla \ldots ) $$

(10)

is a symmetric operator.

$a$ and $b$ are given by

$$ a = \frac{x}{2} \omega \rho_0 \quad , \quad b = \frac{i}{\rho_0} ( \frac{\omega^2}{\mu_0} + \gamma \rho_0 ) $$

(11)

As discussed in Ref. 1 the Hamiltonian of eq. (8) can be written as

$$ H = \frac{i}{2} \{ \gamma, \mathcal{M} \gamma \} $$

(12)

with

$$ \gamma = \begin{pmatrix} \vec{p} \\ \vec{v} \end{pmatrix} \quad , \quad \vec{p} = \vec{v} + \frac{\zeta}{2} \omega $$

(13)

and the symmetric operator

$$ \mathcal{M} = \begin{pmatrix} \mathbb{I} & -E/2 \\ E/2 & F - E/4 \end{pmatrix} $$

(14)

$\gamma$ is expanded in terms of the eigenfunctions of $\mathcal{M}$

$$ \gamma = \sum_{i=1}^{\infty} a_i \gamma_i $$

(15)

where

$$ \gamma_i = \begin{pmatrix} \vec{A} \\ \vec{C} \end{pmatrix} e^{i \vec{k}_i \cdot \vec{r}} $$

(16)
in view of the homogeneity of the equilibrium, \( \vec{A} \) and \( \vec{C} \), as \( \vec{p} \) and \( \vec{v} \), are vectors perpendicular to \( \vec{E}_0 \). The eigenvalues \( \lambda^i \) can be obtained from
\[
M Y^i = \lambda^i Y^i
\]  
which by substitution of eq. (16) in eq. (17) reduces to
\[
\left( \begin{array}{c}
\vec{A}^o + a k_i z^2 y \times \vec{C} \\
-a k_i z^2 y \times \vec{A}^o + (a^2 k_i^2 + b k_i^2) \vec{C}
\end{array} \right) = \lambda^i \left( \begin{array}{c}
\vec{A}^o \\
\vec{C}
\end{array} \right)
\]  
(18)

After crossing the first equation by \( \vec{z} \) the characteristic equation leads to
\[
\lambda^i = \frac{1}{2} \left(1 + a^2 k_i^2 + b k_i^2 \pm \sqrt{(1+a^2 k_i^2+b k_i^2)-4 b k_i^2} \right)
\]  
(19)

For large \( k_i \) the \( \lambda^i \) behave as
\[
\lambda^i \approx a^2 k_i^2
\]  
(20)

and
\[
\lambda^i \approx \frac{b}{a^2} k_i
\]  
(21)

The formula for the expectation values of \( \dot{a}_i^2 \) derived in 1 is
\[
\langle \dot{a}_i^2 \rangle = \frac{1}{\beta \lambda^i}
\]  
(22)

Its application leads here to a spectrum having contributions in \( k_i^2 \) and \( k_i^2 \) for large \( k_i \). The latter contribution is obviously not acceptable. Without gyroviscous effects the contributions would be like
\(-2\kappa^2\) and 1 for large \(\kappa_i\). The latter contribution, though not divergent, is not acceptable either.

This "ultraviolet" catastrophe well known in other areas such as field theory seems to become worse when gyroviscous effects are taken into account. This is true of a Gaussian distribution and is an open question for the full nonlinear problem. Non-Gaussianity may be the key answer as demonstrated for the Korteweg-de Vries equation in 6.

An improvement can also be achieved if the observables are restricted to being functions of \(v^0\) only (not of \(\mathbf{v}^o\)). In this case \(\langle \alpha_i^2 \rangle\) depend on the eigenvalues of \(F\) only, which behave as \(\kappa_i^2\). This is easy to see from the general Hamiltonian introduced in 1:

\[
H = \frac{i}{2} \left[ (v^0 - \frac{\kappa}{2} v^0), (v^0 - \frac{\kappa}{2} v^0) \right] + \frac{1}{2} \left( v^0, F v^0 \right)
\]  

If

\[
\langle f(v^0) \rangle = \frac{\int D(\mathbf{v}^0) D(\mathbf{v}^0) f(\mathbf{v}^0) e^{-\beta H}}{\int D(\mathbf{v}^0) D(\mathbf{v}^0) e^{-\beta H}}
\]  

the integration over \(\mathbf{v}^0\) can be done separately, leaving

\[
\langle f(\mathbf{v}^0) \rangle = \frac{\int D(\mathbf{v}^0) f(\mathbf{v}^0) e^{-\beta H}}{\int D(\mathbf{v}^0) e^{-\beta H}}
\]
For such observables the expectation values are thus unaffected by gyroviscosity and behave as $-\frac{1}{R_c^2}$.

As noticed in 1, the expectation value of the total Hamiltonian diverges for a Gaussian in any case and leads us to the conclusion that non-Gaussianity combined with gyroviscous effects will have to be studied next. This is a very difficult problem in more than one dimension, as discussed in 1.

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References

1 Tasso, H. submitted to Phys. Letter.


5 See, for example,
