On the Mercier Criterion in
Guiding Centre Theory

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Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.
Collisionless two-fluid guiding centre equations are used to investigate stability of toroidal plasma equilibria with respect to localized MERCIER-type modes. The anisotropic equilibria are characterized by \( \sigma(V,B) = 1 - (P_n - P_\perp)/B^2 \), where \( V \) is the volume inside the flux surfaces. It is found that, in general, no localized MERCIER-type eigenmodes exist. For \( \sigma = \sigma(B) \) such modes exist but no explicit stability condition can be given. For \( \sigma = \text{const} \) in the localization region and no trapped particles the necessary stability condition for axisymmetric equilibria agrees with the MERCIER criterion except that the pressure is replaced by \( (P_n + P_\perp)/(2\sigma) \).
1. Introduction

MERCIER (1960) derived a necessary stability condition for axisymmetric toroidal plasmas with respect to displacements which are localized around a rational magnetic surface and are almost constant along the field lines. Subsequently the criterion was shown to be necessary and sufficient with respect to such modes in general toroidal geometry (GREENE and JOHNSON, 1962, MERCIER and LUC, 1974). For axisymmetric equilibria PAO (1975) derived the criterion from an eigenmode analysis.

It is now investigated whether a MERCIER-type mode ansatz may also be used in guiding centre theory (GCT) to derive an analogous stability condition. The GCT (GRAD, 1961) allows for kinetic effects parallel to the field lines and for pressure anisotropy but keeps only $E \times B$ drifts and assumes vanishing guiding centre radii. No collisions are included. GCT-type energy principles were derived by KRUSKAL and OBERMAN (1958) and by ANDREOLETTI (1963). Instead of these principles we use here an eigenmode approach similar to that of PAO (1974, 1975).

Stability and mode analyses of GCT plasmas have hitherto been mainly performed in simple geometries. ALEKSIN and YASHIN (1961), PAO (1974) and CHOE, TATARONIS and GROSSMAN (1977) have investigated modes in straight cylindrical (screw pinch) geometry where in MHD the MERCIER criterion changes to the SUYDAM criterion (SUDDAM, 1958). Stability in bumpy $\theta$-pinch and helical equilibria was investigated by, for example, VAHALA and VAHALA (1977) and WEITZNER (1976) for the cases of small bumpiness and helicity, respectively. Here, we initially consider
arbitrary toroidal plasma geometry. Later on, consideration is restricted to axisymmetric plasmas which are symmetric with respect to the equatorial plane.

The presentation is as follows: Section 2 gives the basic equations. In Section 3 plasma equilibria are described as far as necessary and particular coordinates are introduced. Section 4 presents the linearized eigenmode equations. In Section 5 a MERCIER-type localization ansatz is made and the existence of such modes is discussed. When the pressure anisotropy coefficient \( \sigma = 1 - (P_m - P_t)/B^2 \) is a function of \( B \) only (\( B = |\vec{B}| \)) and the plasma is axisymmetric, the eigenmode equations are further discussed in Section 6. For \( \sigma = \text{const} \) in the localization region and with the further assumption that trapped particles may be neglected a necessary MERCIER-type stability condition is finally derived and compared with the MHD case in Section 7. Conclusions are given in Section 8.
2. Basic Equations

We consider a fully ionized collisionless plasma which is described by the following guiding centre equations (GRAD, 1961):

\[
\rho \frac{d\vec{U}}{dt} = \left[ \vec{j} \times \vec{B} \right] - \nabla \cdot \vec{P},
\]

\[
\vec{j} = \text{curl} \vec{B},
\]

\[
\vec{P} = \vec{P}_{\perp} P_{\perp} + \frac{P_{\parallel} - \vec{P}_{\parallel} \vec{B} B}{B^2},
\]

\[
\frac{dB}{dt} = \text{curl} \left[ \vec{U} \times \vec{B} \right].
\]

The anisotropic pressure \( P_{\parallel}, P_{\perp} \) and the mass density \( \rho \) consist of the contributions from electrons and ions:

\[
P_{\parallel, \perp} = P_{\parallel, \perp}^+ + P_{\parallel, \perp}^-,
\]

\[
\rho = \rho^+ + \rho^-,
\]

which, in turn, are defined as moments of the distribution functions \( F^\pm \) of the guiding centres:

\[
\left[ P_{\parallel, \perp}^\pm \right] = \int dv \int d\mu \left[ v^2, \mu B \right] F^\pm(v, \mu, \vec{r}, t),
\]

\[
\rho^\pm = m^+ n^+ = \int dv \int d\mu F^\pm,
\]

where \( v \) is the velocity of the guiding centres along the field lines, and \( \mu \) their magnetic moment. The \( F^\pm \) are determined from the kinetic equations.
\[
\frac{\partial F^\pm(v, u, r, t)}{\partial t} + v \cdot \left[ (\mathbf{u} + v\mathbf{u}) F^\pm \right] + \\
+ \frac{3}{\partial v} \left\{ \left( \frac{e^-}{m^-} E_n + \mathbf{\beta} \cdot \nabla \left( \frac{u^2}{2} + m\mathbf{B} \right) + v \mathbf{\kappa} \cdot \mathbf{u} \right) F^\pm \right\} = 0 ,
\]

(2.7)

where \( e^\pm \) and \( m^\pm \) are the charge and mass of the ions and electrons, respectively, and

\[
\mathbf{\beta} = \frac{\mathbf{B}}{B}, \quad \mathbf{\kappa} = (\mathbf{\beta} \cdot \nabla) \mathbf{\beta} ,
\]

(2.8)

\[
\mathbf{u} = \mathbf{U} - \mathbf{\beta} (\mathbf{U} \cdot \mathbf{\beta}) = \left[ \mathbf{\beta} \times \left[ \mathbf{U} \times \mathbf{\beta} \right] \right] .
\]

(2.9)

The electric field parallel to the magnetic field lines,

\[
E_n = -\mathbf{\beta} \cdot \nabla \phi ,
\]

(2.10)

is given by

\[
E_n + \sum_{-} \left( \frac{e^-}{m^-} \right)^\pm + \sum_{+} \left( \frac{e^+}{m^+} \right)^\pm = \mathbf{\beta} \cdot \nabla \mathbf{P}^\pm
\]

(2.11)

or by the assumption of quasineutrality, \( \sum_{-}^\pm (en)^\pm = 0 \), which we shall adapt here.

For later purposes it is useful to write the equation of motion (2.1) in two alternative forms (NORTHROP and WHITEMAN, 1964):

\[
\rho \frac{d\mathbf{u}}{dt} = -\nabla (P_n + \frac{B^2}{2}) + (\mathbf{\beta} \cdot \nabla) \mathbf{\phi} \]

(2.12)

\[
= \left[ \mathbf{K} \times \mathbf{\beta} \right] - \nabla P_n + \frac{P_n - P_1}{B^2} \nabla \mathbf{B} ,
\]

(2.13)

where

\[
\mathbf{K} = \text{curl} \mathbf{\phi} \mathbf{B}
\]

(2.14)
and

$$\sigma = 1 - \frac{P_\parallel - P_\perp}{B^2}$$  \hfill (2.15)$$

($\sigma > 0$ is assumed to avoid fire hose instability; see, for example, KADISH (1966)).
3. Equilibria and coordinates

For static equilibria we have from equ. (2.13)

\[
\left[ \hat{k} \times \hat{B} \right] = \nabla P_n - \frac{P_n - P_\perp}{B^2} \nabla B .
\] (3.1)

A discussion of anisotropic equilibria which satisfy equ. (3.1) is given by, for example, SPIES and NELSON (1974). We assume that the plasma has arbitrary toroidal geometry with nested joint flux surfaces of \( \hat{B} \) and \( \hat{k} \).

It is also assumed that \( P_n \) is given as an arbitrary function of the flux surfaces and the absolute values of \( \hat{B} \):

\[
P_n = P_n(V,B),
\] (3.2)

where \( V \) is the volume contained inside each flux surface.

It follows from equus. (3.1) and (3.2) that

\[
P_\perp = P_n - B \frac{\partial P_n}{\partial B}.
\] (3.3)

if \( \hat{B} \cdot \nabla B \neq 0 \) (which excludes circular straight cylinders, i.e. screw pinch geometry). Hence, \( P_\perp(V,B) \) and \( \sigma(V,B) \) are completely determined from \( P_n(V,B) \). For the pressure balance one obtains

\[
\left[ \hat{k} \times \hat{B} \right] = P_n'(V,B)\nabla V .
\] (3.4)

For the derivative with respect to \( V \) the following notational convention is adopted: \( A' = (\partial / \partial V)|_B A \), \( \hat{A} = (\partial / \partial V)|_{r^2, r^3} A \), where \( r^2, r^3 \) are
arbitrary nonsingular coordinates on the flux surfaces. For isotropic pressure, $P_n = P_\perp$, one has $\partial P_n / \partial B = 0$, $\sigma = 1$ and $\vec{K} = \vec{J}$.

It is easily seen that static equilibrium distribution functions are given by

$$ F^\pm(\nu, \mu, \vec{r}) = BF^\pm(\nu^\pm, \mu, \vec{r}) ,$$

(3.5)

where $\nu^\pm = \nu^2 / 2 + \mu B + (e/m)^{\pm} \phi$ and $F^\pm$ are arbitrary functions of $\nu^\pm, \mu, \vec{r}$, provided that $\vec{B} \cdot \nabla |_{\nu^\pm, \vec{r}} = 0$. By taking moments of these distribution functions it follows that equ. (3.3) is valid separately for each species. With $\vec{B} \cdot \nabla \cdot \vec{P}^\pm = \vec{B} \cdot \nabla P_n^\pm + (P_\perp - P_n) \vec{B} \cdot \nabla B / B$ one then obtains $\vec{B} \cdot \nabla \cdot \vec{P}^\pm = 0$ so that according to equ. (2.10), (2.11) $E_n = 0$, $\phi = \text{const}$ along $\vec{B}$. One may set $\phi = 0$ without loss of generality. Toroidal equilibria without electric field exist in guiding centre theory because only $\vec{E} \times \vec{B}$ drifts common to all species are retained.

For the calculation of axisymmetric equilibria it is useful to specify $r^2 = \tilde{\theta}$, $r^3 = \tilde{\phi}$, where $\tilde{\theta}$ is an arbitrary poloidal variable and $r, \tilde{\phi}, z$ are cylindrical coordinates with $\tilde{\phi}$ equal to the ignorable angle. Explicit representation of $\vec{B}$ and $\vec{K}$ in terms of the poloidal fluxes $\chi(V)$, $I(V)$ of $\vec{B}$ and $\vec{K}$ is then possible:

$$ \vec{B} = \frac{1}{2\pi} \left\{ [\nabla \tilde{\phi} \times \nabla \chi] + \Lambda(\chi, \tilde{\theta}) \nabla \tilde{\phi} \right\} ,$$

$$ \vec{K} = \frac{-1}{2\pi} \left\{ [\nabla \tilde{\phi} \times \nabla \Lambda] - r^2 \text{div} \left( \frac{\sigma \nabla \chi \times \nabla \tilde{\phi}}{r^2} \right) \right\} ,$$

(3.6)

with $\sigma \Lambda = I(V_\perp) - I(V)$. The flux surfaces $\chi = \text{const}$ are determined by

$$ \text{div} \left( \frac{\sigma}{r^2} \nabla \chi \right) + \frac{1}{r^2} \Lambda \frac{d\sigma \Lambda}{d\chi} = -4\pi^2 \frac{1}{\chi} P_n(V, B) .$$

(3.7)
In general we shall use covariant notation with coordinates $r^\alpha$, $\alpha = 1, 2, 3$, where $r^1 = V$, and where $r^m$, $m = 2, 3$, are arbitrary coordinates on the flux surfaces. The summation convention is that Greek indices indicate summation over all indices, while Latin indices indicate summation over surface coordinates only. Some of the usual definitions and relations are (LAUGWITZ, 1960)

$$A^\alpha = \dot{\alpha} \cdot \nu r^\alpha, \quad A_\alpha = \dot{\alpha} \cdot e_\alpha, \quad \dot{e}_\alpha = \frac{\partial r^\alpha}{\partial r^\alpha},$$

$$g_{\alpha\beta} = e_\alpha \cdot e_\beta, \quad A_\alpha = g_{\alpha\beta} A^\beta,$$

$$\left[ \frac{\dot{A} \times \dot{B}}{B} \right]_\alpha = \frac{1}{h} \epsilon_{\alpha\beta\gamma} A^\beta B^\gamma,$$

$$\text{div} \ \dot{A} = h \frac{\partial}{\partial r^\alpha} \left( \frac{1}{h} A^\alpha \right),$$

$$\left( \text{curl} \ \dot{A} \right)^\alpha = h \epsilon^{\alpha\beta\gamma} \frac{\partial A_\gamma}{\partial r^\beta},$$

$$h = \left[ \nu r^1 \times \nu r^2 \right] \cdot \nu r^3 = \sqrt{\det (g^{\alpha\beta})}.$$

Coordinates derived from SPIES-NELSON coordinates are particularly useful for the analysis of localized eigenmodes. For anisotropic equilibria SPIES and NELSON (1974) proved the existence of coordinates $r^\alpha$ with the properties: $r^1 = V$, $r^2$ and $r^3$ are poloidal-like and toroidal-like coordinates with periodicity unity and

$$B^1 = 0, \quad B^2 = h\dot{X}, \quad B^3 = h\dot{\psi},$$

$$K^1 = 0, \quad K^2 = h\dot{I}, \quad K^3 = h\dot{J},$$

(3.9)
where $\psi(V)$, $J(V)$ are the toroidal fluxes of $\vec{B}$, $\vec{K}$. The Jacobian $h(V,r^2,r^3)$ is normalized by $\int_{-1}^{1} \int_{-1}^{1} dr^2 dr^3 h^{-1} = 1$. The pressure balance is

$$h(\ddot{\psi} - \ddot{J}_\chi) = P'_n(V,B). \quad (3.10)$$

The safety factor is defined as $q = B^3 / B^2 = \dot{\psi} / \dot{\chi}$. For isotropic pressure one gets $h = 1$ and the coordinates change to HAMADA coordinates.

Let $V = V_\circ$ be a rational surface where $q_\circ = q(V_\circ) = M/N$. The coordinates $\theta, \phi$ are defined by $\theta = r^2$, $\phi = r^3 - q_\circ r^2$, which implies that

$$B^\theta = h_\chi \dot{\chi}, \quad B^\phi = (q - q_\circ) h_\chi \dot{\chi},$$
$$K^\theta = h I, \quad K^\phi = (q - q_\circ) h I - \frac{P'_n}{\chi} \quad (3.11)$$

and that

$$\vec{B} \cdot \nabla = \dot{h}_\chi \left[ \frac{\partial}{\partial \theta} + (q - q_\circ) \frac{\partial}{\partial \phi} \right], \quad (3.12)$$
$$\vec{K} \cdot \nabla = \dot{h} I \left[ \frac{\partial}{\partial \theta} + (q - q_\circ) \frac{\partial}{\partial \phi} \right] - \frac{P'_n}{\chi} \frac{\partial}{\partial \phi}.$$

For $q = q_\circ$ $\theta$ is a coordinate along $\vec{B}$ and $\phi$ counts the field lines. Periodicity conditions are

$$a(\theta, \phi) = a(\theta, \phi + 1) = a(\theta + N, \phi) = a(\theta + 1, \phi - \frac{M}{N}) \quad (3.13)$$
4. Eigenmode equations

The system of eigenmode equations is obtained by linearizing equs. (2.1) to (2.9) around static equilibria of the type discussed in the last section. The time dependence of the linearized quantities is assumed to be of the form $e^{-i\omega t}$. The first-order quantities $\vec{B}^{(1)}$, $P^{(1)}_{\alpha \beta}$, $\phi^{(1)}$, $\phi^{(1)}$ will be denoted by $\vec{b}$, $P_{\alpha \beta}^{\pm}$, $f^{\pm}$, $\phi$. The displacement vector $\vec{\xi}$ is defined by $\vec{\xi}^{(1)} = \frac{\partial \vec{\xi}}{\partial t} = -i\omega \vec{\xi}$. Its surface component is decomposed into the components $x$ and $y$ along the equilibrium fields $\vec{B}$ und $\vec{K}$: $\vec{\xi} = x \vec{B} + y \vec{K} + \vec{\xi}_{\text{normal}}$. In covariant notation

$$\xi^m = x B^m + y K^m.$$  \hspace{1cm} (4.1)

From equs. (2.4), (3.4) and $\text{div} \vec{B} = 0$ one obtains

$$b^\alpha = B^m \frac{\partial \xi^\alpha}{\partial r^m} - \xi^\beta \frac{\partial B^\alpha}{\partial r^\beta} - B^\alpha \text{div} \xi$$  \hspace{1cm} (4.2)

so that

$$b^1 = D\xi^1$$  \hspace{1cm} (4.3)

and, with equ. (4.1),

$$b^m = B^m D_x + K^m D_y + T^m - B^m \xi - B^m \text{div} \xi.$$  \hspace{1cm} (4.4)

Here and in the following we shall use the definitions

$$D = \vec{B} \cdot \nabla = B^m \frac{\partial}{\partial r^m}, \quad E = \vec{K} \cdot \nabla = K^m \frac{\partial}{\partial r^m}.$$  \hspace{1cm} (4.5)
The vector \( \hat{T} \) is defined by \( \hat{T} = D\hat{K} - E\hat{B} \), which, using \( \text{div} \, \hat{B} = \text{div} \, \hat{K} = 0 \) and equs. (2.15), (3.3), (3.4), may be expressed as

\[
\hat{T} = \text{curl} \left[ \hat{K} \times \hat{B} \right] = \left[ \nabla p_i \times \nabla v \right] = \frac{\partial p_i}{\partial b} \left[ \nabla b \times \nabla v \right] \nonumber \\
= B \left[ \nabla \sigma \times \nabla b \right]. \tag{4.6}
\]

For isotropic plasmas one gets \( \hat{T} = 0 \).

Let

\[
p^* = \left( p_\perp + \frac{1}{2} B^2 \right)(1) = p_\perp + \hat{B} \cdot \hat{b}. \tag{4.7}
\]

In the equation of motion (2.12) \( p^* \) is the only linearized quantity which has a derivative in the radial direction. Using equs. (4.3), (4.4), the linearized covariant component of equ. (2.12) with index 1 yields

\[
\frac{\partial p^*_1}{\partial r_1} = (D\sigma B_1 + a)\chi + (D\sigma K_1 D + bD + \tau_1)\gamma 
\nonumber \\
+ \rho \omega^2 (B_1 \chi + K_1 \gamma + g_{11} \xi^1) 
\nonumber \\
+ (Dg_{11} D + A + c_1 D) \xi^1 
\nonumber \\
+ (DB_1 + s_1) \sigma(1), \tag{4.8}
\]

where

\[
a = P_i + \sigma B_m B^m, \quad b = \sigma K_m B^m,
\]

\[
A = -Dg_{1m} B^m \sigma_{mn} B^m + \frac{\partial B^m}{\partial r} \left( \frac{\partial B}{\partial r} - \frac{\partial B_1}{\partial r} \right) - \sigma B_m B^m, \tag{4.9}
\]

\[
-\sigma_B B^m, \quad
\]
\[ c_1 = \sigma \dot{B}_1, \quad s_1 = \frac{1}{2} (B_m \dot{B}_m^m - \dot{B}_m^m) \quad (4.9) \]

\[ \tau_1 = T_m (\sigma B_m^m)' - T_m \left( \frac{\partial \sigma B_m}{\partial r^1} - \frac{\partial \sigma B_m}{\partial r^m} \right) + D \sigma T'_1, \]

and

\[ \chi = D x - \text{div} \dot{\xi}. \quad (4.10) \]

Here and in the following the operators \( D \) and \( E \) act on all quantities to the right of them, except if enclosed in square brackets \([ ]\). The only other radial derivative in the eigenmode equations occurs in the definition of \( \text{div} \dot{\xi} \) which using equs. (3.8) and (4.1), may be solved for

\[ \frac{\partial \xi_1}{\partial r^1} : \]

\[ \frac{\partial \xi_1}{\partial r^1} = -\chi - Ey - h \left( \frac{\partial}{\partial r^1} \frac{1}{h} \right) \xi_1. \quad (4.11) \]

The remaining covariant components of equ. (3.8), when contracted with \( \ddot{\dot{B}} \) and \( \ddot{\dot{K}} \), yield

\[ (D \sigma \dot{\dot{B}} + a_1) X + (D \sigma \dot{\dot{K}} D + b_1 D + D \tau_2 + \tau_3) y \]

\[ + \rho \omega^2 (\ddot{\dot{B}} \cdot \dot{x} + \ddot{\dot{B}} \cdot \dot{K} y) = \quad (4.12) \]

\[ = (-D \sigma B_1 D + Da + c_2 + c_3 D - \rho \omega^2 B_1) \xi_1 \]

\[ + (-D \ddot{B} \cdot \ddot{B} + s_2) \sigma^{(1)} + D \rho^* \]

and

\[ (D \sigma \ddot{\dot{B}} + a_2) X + (D \sigma \ddot{\dot{K}} D + b_2 D + D \tau_4 + \tau_5) y \]

\[ + \rho \omega^2 (\ddot{\dot{B}} \cdot \ddot{K} x + \ddot{\dot{K}} \cdot \ddot{K} y) = \quad (4.13) \]

\[ = (-D \sigma K_1 D + Db + c_4 + c_5 D - \rho \omega^2 K_1) \xi_1 \]

\[ + (-D \ddot{B} \cdot \ddot{K} + s_3) \sigma^{(1)} + E \rho^* \]
The coefficients are defined as follows

\[ a_1 = \left[ D\sigma \right] \hat{B} \cdot \hat{B}, \quad a_2 = \left[ E\sigma \right] \hat{B} \cdot \hat{B} - \sigma \hat{T} \cdot \hat{B}, \]
\[ b_1 = \left[ D\sigma \right] \hat{B} \cdot \hat{K}, \quad b_2 = \left[ E\sigma \right] \hat{B} \cdot \hat{K} - \sigma \hat{T} \cdot \hat{K}, \]
\[ c_2 = \left[ D\sigma \right] B_m \hat{B}^m - DP, \quad c_3 = -\left[ D\sigma \right] B_1, \]
\[ c_4 = \left[ E\sigma \right] \frac{\varepsilon}{\varepsilon^r} B_m \hat{B}^m - \sigma \hat{T} \hat{m} \hat{m}, \quad c_5 = -\left[ E\sigma \right] B_1 + \sigma T_1, \]
\[ s_2 = \frac{1}{2} \frac{D}{E} B_1, \quad s_3 = \frac{1}{2} \frac{E}{B} \hat{B} \cdot \hat{B} + \frac{\sigma}{\hat{T}} \hat{B}, \]
\[ \tau_2 = \sigma \hat{B} \cdot \hat{T}, \quad \tau_3 = \left[ D\sigma \right] \hat{B} \cdot \hat{T}, \]
\[ \tau_4 = \sigma \hat{K} \cdot \hat{T}, \quad \tau_5 = \left[ E\sigma \right] \hat{B} \cdot \hat{T} - \sigma \hat{T} \cdot \hat{T}. \]

The l.h.s. sides of eqns. (4.12), (4.13) constitute first-order differential operators along \( \hat{B} \) for \( X \), and second-order operators for \( y \).

It remains to determine \( \sigma^{(1)} \) and \( \text{div} \, \xi \) as functions of the other linearized quantities. Equations (2.15), (4.7) yield

\[ \sigma^{(1)} = \frac{1}{B} \left( \int_{-\infty}^{+\infty} \phi \, dv \, \frac{\varepsilon^r}{\varepsilon^2} \, du \, f^\pm \right), \]

(4.15)

where the partial pressures

\[ p^\pm_{\|} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (B^{(1)} F^\pm + B f^\pm), \]

(4.16)

yield \( p^\pm_{\|} = p^{\|, \pm} + p^{\perp, \pm} \), with \( B^{(1)} = \hat{B} \cdot \hat{b}/B = (p^* - p_{\perp})/B \).
From the equation of continuity which follows from equ. (2.7) one obtains

\[
\text{div} \ 1 \ \xi = -\frac{1}{\rho} \ (\xi \cdot \nabla \rho + \xi \int_0^\infty \int_{\infty}^{0} \ dv \ f \ d\mu f^\pm) . \tag{4.17}
\]

Quasineutrality of the displacement requires that

\[
\int_0^\infty \int_{\infty}^{0} \ dv \ f \ d\mu f^\pm = 0 . \tag{4.18}
\]

The functions \( f^\pm \) are determined from equ. (2.7). With the definitions \( \varepsilon = v^2/2 + \mu B \) and \( f^\pm = B f^\pm \) one gets

\[
\left( \frac{\partial}{\partial t} + \frac{v}{B} D \right) f^\pm (\varepsilon, \mu, r, t) =
\]  
\[= C_1 F^\pm + C_2 \cdot v |F^\pm| + C_3 \frac{\partial F}{\partial v} , \tag{4.19}
\]

where

\[
C_1 = -\frac{1}{B} \left[ \frac{\partial}{\partial t} \ (\text{div} \ \eta + \kappa \cdot \eta) + v \ \text{div} \ \beta^{(1)} \right] ,
\]

\[
C_2 = -\frac{1}{B} \left[ \frac{\partial \eta}{\partial t} + v \beta^{(1)} \right] , \tag{4.20}
\]

\[
C_3 = \frac{1}{B} \left[ \frac{\partial}{{\mu}} + \frac{1}{B} D\phi + \mu (\beta^{(1)} \cdot \nabla B + \beta \cdot \nabla B^{(1)} - v \frac{\partial \kappa \cdot \eta}{\partial t} \right] \tag{4.21}
\]

and

\[
\eta = \left[ \beta \times \left[ \frac{\partial}{\partial t} \right] \right] \tag{4.21}
\]

The ansatz

\[
f^\pm = \xi \cdot F^\pm - \eta \cdot v |F^\pm| + \left[ \frac{\partial}{{\mu}} \phi + \mu B \xi \right] \frac{\partial F^\pm}{\partial \varepsilon} + Bg , \tag{4.22}
\]
where
\[ \zeta \equiv - (\text{div} \, \nabla + \kappa \cdot \eta) = \frac{1}{\mathfrak{B}} \left( \eta \cdot \nabla \mathfrak{B} + \frac{p^* - p_1}{\mathfrak{B}} \right) \] (4.23)

separates \( f^\pm \) into an adiabatic part and an \( \omega \)-dependent part \( \tilde{g}^\pm \), which after some algebraic manipulation is found to satisfy the equation
\[ (-i\omega + \frac{v}{\mathfrak{B}} D) \tilde{g}^\pm = i\omega \psi^\pm \frac{3F^\pm}{\mathfrak{B}} , \] (4.24)

where
\[ \psi^\pm = \left( \frac{e}{m} \right)^\pm \phi + \mu B \zeta + \kappa \cdot \eta \cdot v^2 \] (4.25)

and \( v^2 = 2(\varepsilon - \mu B) \). The second half of equ. (4.23) follows from the definition of the curvature \( \kappa \) and equ. (4.2), (4.7), (4.21). Terms containing \( \eta \) may be expressed as functions of \( \xi \) by the relations
\[ \eta^1 = \xi^1 , \quad \kappa \cdot \eta = \kappa \xi^1 + \kappa \cdot \eta \cdot y , \] (4.26)
\[ \eta \cdot \nabla \mathfrak{B} = \xi^1 (B - B(DB)) + y(\kappa - \frac{\vec{\mathfrak{B}}}{B^2}) \cdot \nabla \mathfrak{B} . \]

The component \( x \) of \( \xi \) in the direction along \( \vec{B} \) does not occur in the microscopic equations.

In the formation of density and pressure moments from \( f^\pm \) it is useful to have relations between the quantities \( A^\pm_{m,n} \) and \( B^\pm_{m,n} \), defined by
\[ A^\pm_{m,n} = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{\infty} d\mu (\mu B)^m v^{2n} F^\pm(\varepsilon, \mu, \vec{r}) , \]
\[ B^\pm_{m,n} = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{\infty} d\mu (\mu B)^m v^{2n} \frac{\partial F^\pm}{\partial \varepsilon} . \] (4.27)
Direct verification and partial integration yields

\[ B_{m,n}^\pm = (-m + B \frac{\partial}{\partial B}) A_{m-1,n}^\pm = -(2n - 1) A_{m,n-1}^\pm, \]  

(4.28)

where the second equality is only valid for \( n > 0 \). Quasineutrality of the equilibrium implies that

\[ \sum_{n=-\infty}^{\infty} \left( \frac{e}{m} \right)^\pm A_{0,n}^\pm = 0. \]  

(4.29)

With equs. (4.28), (4.29) quasineutrality of the linearized displacements requires that

\[ \Phi \sum_{n=-\infty}^{\infty} \left( \frac{e}{m} \right)^\pm B_{0,n}^\pm + \sum_{n=-\infty}^{\infty} \left( \frac{e}{m} \right)^\pm \int_0^\infty dv \int_0^\infty d\mu \frac{\partial}{\partial \mu} B_{v}^\pm = 0. \]  

(4.30)

Similarly, the pressure moments are determined by

\[
\begin{bmatrix}
 p^\parallel \\
 p^\perp
\end{bmatrix} = -\xi \begin{bmatrix}
 p^\parallel \\
 p^\perp
\end{bmatrix} + \frac{p^\star - p^\perp}{B} \begin{bmatrix}
 \partial p^\parallel / \partial B \\
 \partial p^\perp / \partial B
\end{bmatrix} + \\
+ \sum_{n=-\infty}^{\infty} \int_0^\infty dv \int_0^\infty d\mu \left[ \frac{\nu^2}{\mu B} \right] B_{v,n}^\pm
\]

(4.31)

and \( \text{div} \bar{\xi} \) is given by

\[
-\rho \text{div} \bar{\xi} = (\bar{\xi} \cdot \nabla B + \frac{p^\star - p^\perp}{B} ) \frac{\partial \rho}{\partial B} + \Phi \sum_{n=-\infty}^{\infty} \left( \frac{e}{m} \right)^\pm B_{0,n}^\pm
\]

(4.32)

\[
+ \sum_{n=-\infty}^{\infty} \int_0^\infty dv \int_0^\infty d\mu \frac{\partial}{\partial \mu} B_{v,n}^\pm.
\]

In order to obtain eigenmodes and eigenvalues \( \omega \), the following problems would have to be solved step by step. The differential equation for \( g^\pm \) along \( \hat{B} \) is solved as a path integral over source terms.
containing $p_\perp$, $\phi$, $y$, $\xi^1$, $\rho^*$. The solution, inserted into the quasi-
neutrality condition (4.30) and the equation (4.31) for the pressure $p_\perp$
yields two coupled integral equations of the second kind for $\phi$ and $p_\perp$.
Their solution and, from equs. (4.31), (4.32), (4.13), also $p_{\nu}$, $\text{div} \xi$
and $\sigma^{(1)}$ may be expressed as convolutions over $y$, $\xi^1$, $\rho^*$ (MICHLIN, 1962).
In the next step the integro-differential equations (4.12), (4.13) for
$x$ and $y$ are solved as functionals of $\xi^1$, $\rho^*$, and the solutions are in-
serted into the equs. (4.8), (4.11) with the radial derivatives for $\xi^1$, $\rho^*$.
The solution of these equations together with periodicity and
boundary conditions finally determines the eigenvalues.

It is obvious that no general explicit solution is possible. In
the next section a particular class of eigenmodes will be further in-
vestigated.
5. Localized Mercier-type eigenmodes

In MHD the Mercier criterion may be obtained by considering modes
(MERCIER and LUC, 1974; PAO, 1975) which are confined in a small region
$\Delta V$ around a rational surface $V = V_0$,

$$\left| \frac{\Delta V}{V_0} \right| = \varepsilon << 1 , \quad (5.1)$$

and have frequencies $\omega = O(\varepsilon)$ . With the definition $\omega = \varepsilon \omega$ this amounts
to $\omega = O(1)$. (No confusion should arise here with the previously defined
$\varepsilon = v^2/2 + uB$.) Introducing the scaled radial coordinate $s = (V - V_0)/\varepsilon$ ,
with $\partial / \partial V = (\partial / \partial s)/\varepsilon$ , the localization is expressed by $s = O(1)$ .

All nonequilibrium quantities are expanded in powers of $\varepsilon$, starting
(arbitrarily) with $\varepsilon^{-1}$:

$$x = \frac{1}{\varepsilon} x_{-1} + x_0 + x_1 + \cdots , \quad (5.2)$$

$$o(1) = \frac{1}{\varepsilon} \sigma_{-1} + \sigma_0 + \varepsilon \sigma_1 + \cdots ,$$

e tc. It follows from equs. (4.8), (4.11) that $p^* = p_0^* + \varepsilon p_1^* + \cdots$ and
$\xi_1^* = \xi_0 + \varepsilon \xi_1 + \cdots$ cannot have a $O(\varepsilon^{-1})$ contribution owing to $\partial p/\partial s = 0$
and $p^* = 0$ on the boundary of the localization domain.

The operators $D$ and $E$ are expanded as follows:

$$D = h \chi \left[ \frac{\partial}{\partial \theta} + \varepsilon q s \frac{\partial}{\partial \phi} + \cdots \right] , \quad (5.3)$$

$$E = h \chi \left[ \frac{\partial}{\partial \theta} + \varepsilon q s \frac{\partial}{\partial \phi} + \cdots \right] - \frac{p_1}{\chi} \frac{\partial}{\partial \phi}$$
In lowest order one obtains from equs. (4.12), (4.13)

\[(D_o \sigma \vec{B} \cdot \vec{B}) X_{-1} + (D_o \sigma \vec{K} \cdot \vec{K}D_o + b_1D_o + D_o \tau_2 + \tau_3)y_{-1}\]

\[= (D_o \sigma \vec{B} \cdot \vec{B} + s_2)\sigma_{-1},\]

\[(D_o \sigma \vec{B} \cdot \vec{K} + a_2) X_{-1} + (D_o \sigma \vec{K} \cdot \vec{K}D_o + b_2D_o + D_o \tau_4 + \tau_5)y_{-1}\]

\[= (D_o \sigma \vec{B} \cdot \vec{K} + s_3)\sigma_{-1},\]

where \(D_o = \hbar \frac{\partial}{\partial \theta}.\) Equation (4.24) yields \(D_o \sigma_{-1} = 0\) so that \(\sigma_{-1} = \sigma_{-1}(v, \phi).\) This function is determined by going to the next order in \(\varepsilon,\) some details of which will be touched on in Section 6. Here, it suffices to know that following the procedure described in the last section \(\sigma_{-1}(y_{-1})\) is eventually determined as a convolution over \(y_{-1}\) (although no explicit representation of the kernel exists in general).

As a result, equs. (5.4) are a coupled system of integro-differential equations for \(X_{-1}\) and \(y_{-1}\) with a free parameter \(w\) which enters the equations only because \(\sigma_{-1} = \sigma_{-1}(v, y_{-1}).\) As in the theory of differential equations with periodic coefficients (KAMKE, 1977), it may be shown that the periodicity condition in \(\theta\) for \(X_{-1}\) and \(y_{-1}\) here also fixes the eigenvalues \(w_{n\sigma}\) and eigenfunctions \(X_{-1} = X_{n\sigma}(\theta), y_{-1} = y_{n\sigma}(\theta).\) This spectrum does not exist in MHD where \(\sigma(1) = 0.\) Its eigensolutions have nothing in common with MERCIER-type modes where eigenvalues are determined by a radial matching procedure (PAO (1974), (1975)). Here, the \(w_{n\sigma}\) spectrum will not be considered any further.

In order to recover MERCIER-type modes, it is obviously necessary that \(w\) should not enter at this stage so that it is not "prematurely"
fixed. As a first requirement, this restricts one to modes \( y_{-1} \) and equilibria such that \( \sigma_{-1} = 0 \). The only periodic solution of the resulting differential equations (5.4) with periodic coefficients is, however, in general, the trivial solution \( x_{-1} = y_{-1} = 0 \). To dominant order, this is a degenerate mode with displacements only along \( \hat{\mathbf{B}} \), \( x_{-1} \neq 0 \), but not a MERCIER mode with \( y_{-1} \neq 0 \).

In order to obtain MERCIER modes, as a second condition, the coefficients of \( y_{-1} \) in equus. (5.4) have to vanish. This implies

\[
\hat{\mathbf{T}} = [\nabla \sigma \times \nabla \mathbf{B}] = 0
\]  

(5.5)

so that \( \tau_i = 0 \), \( i = 1, \ldots, 5 \). In this case the "trivial" periodic solution is

\[
x_{-1} = \hbar \frac{\partial x_{-1}}{\partial \theta} - (\text{div} \ \hat{\mathbf{T}})_{-1} = \frac{\partial y_{-1}}{\partial \theta} = 0 ,
\]

(5.6)

which allows \( x_{-1} \), \( y_{-1} \neq 0 \) and is the analogue of the MHD case for which \( \partial x_{-1}/\partial \theta = (\text{div} \ \hat{\mathbf{T}})_{-1} = \partial y_{-1}/\partial \theta = 0 \) (MERCIER and LUC, 1974).

Equation (5.5) implies the requirement \( \sigma(V,B) = \sigma(B) \) and is equivalent to

\[
P_n(V,B) = P_1(V) + P_2(B)
\]  

(5.7)

with arbitrary functions \( P_1 \) and \( P_2 \).

The pressure balance becomes \( [\hat{\mathbf{R}} \times \hat{\mathbf{B}}] = \nabla P_1 \), which is completely analogous to the MHD equation. This correspondence together with \( \sigma_{-1} = 0 \) makes plausible the existence of MHD-like eigenmodes in this case.
On the other hand, for general $\sigma = \sigma(V, B)$, the plasma anisotropy seems to make the plasma so "stiff" that either $\sigma_{-1}$ is nonzero or the frequency of the eigenmodes is higher, $\omega = O(\varepsilon^0)$, or the anisotropy does not allow $x, y, \xi^1$ to be of different order, in contrast to the localization assumption $x, y = O(\varepsilon^{-1})$, $\xi^1 = O(\varepsilon^0)$.

For the present purposes this restricts one in the following to equilibria with $\sigma = \sigma(B)$. In order to cope more easily with the requirement $\sigma_{-1} = 0$, we shall, in addition, restrict ourselves to axisymmetric (tokamak) equilibria which are symmetric with respect to the equatorial plane.
6. Modes in axisymmetric \( \sigma(B) \) equilibria

For axisymmetric equilibria \( \phi \) is an ignorable coordinate and the ansatz
\[ e^{2\pi n s} \] for the linearized quantities can be made. Equations (4.24)
can then be written as

\[
\frac{\partial g^{\pm}}{\partial \theta} + \epsilon \lambda g = \epsilon G^{\pm},
\]

\[
\lambda = \lambda_o + \epsilon \lambda_1 + \cdots = (2\pi n qs - \frac{\omega_B}{v \chi}) i + \epsilon i \pi n qs^2 + \cdots ,
\]

\[
G^{\pm} = \frac{i \omega_B}{v \chi} \psi \frac{\partial F^{\pm}}{\partial \epsilon}
\]

\[
= \frac{1}{\epsilon} G^{\pm}_{-1} + G^{\pm}_0 + \cdots .
\]

In order to determine \( g^{\pm}_{-1} \), \( g^{\pm}_0 \), etc., a distinction between trapped and
untrapped particles has to be made. Boundary conditions for untrapped
particles are \( g^{\pm}(\theta) = g^{\pm}(\theta + N) \), while for trapped particles \( g^{\pm}(\theta_i, \text{sign}
\nu) = g^{\pm}(\theta_i, -\text{sign} \nu) \) states that no particles should be lost at the
turning points \( \theta_i \), \( i = 1,2 \). (Generalization to more than one pair
of turning points is obvious.) From \( \nu^2 = 2[\epsilon - \mu B(\theta)] \) there follow
the domains (I) untrapped: \( 0 < \mu < \epsilon / B_{\max} \), and (II) trapped:

\[ \frac{\epsilon}{B_{\max}} < \mu < \epsilon / B, \] where \( B_{\max} \) is the maximum of \( B(\theta). \) The turning points
are determined by \( \mu B(\theta_i) = \epsilon. \)

One easily obtains for untrapped and trapped particles

\[
g^{\pm}_{-1u} = \frac{<G^{\pm}_{-1}>}{<\lambda_o>}; g^{\pm}_{-1t} = \frac{<G^{\pm}_{-1}>}{<\lambda_v>}
\]

\[
(6.2)
\]

where

\[
<A>_{\theta} = \frac{1}{N} \int_{\theta_o}^{\theta} \int_{\theta} A(\theta) d\theta, \quad <A>_{\theta_1} = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \int_{\theta_1} A(\theta) d\theta
\]

\[
(6.3)
\]
and \( \lambda_v = -i\omega B \). 

From the assumed up/down symmetry of the equilibria it follows that terms with \( y_{-1} \) vanish identically in \( \langle C^+_{-1} \rangle \) and \( \langle C^-_{-1} \rangle \). This follows from equs. (4.23) to (4.26) and the fact that \( \hat{\tau} \cdot \kappa, \hat{\tau} \cdot \nu \) and \( \hat{\nu} \cdot \nu \) are antisymmetric, while \( \hat{\nu} \cdot \nu \) and \( y_{-1} \) are symmetric. (\( y_{-1} \) is constant.) The integral equations for \( p_{1,-1} \) and \( \phi_{-1} \), obtained by inserting equs. (6.2) into equs. (4.30), (4.31) are therefore purely homogeneous. The trivial solution \( p_{1,-1} = \phi_{-1} = 0 \) is acceptable (at least for \( \beta \ll 1 \), \( s \neq 0 \) it is even the only solution; see Appendix A) and agrees with the MHD ordering. The result so far is then \( X_{-1} = \partial y_{-1} / \partial \theta = p_{n_{-1}} = p_{1,-1} = \phi_{-1} = \sigma_{-1} = 0. \)

From equs. (4.11), (4.8) one obtains

\[
\frac{\partial \xi_o}{\partial s} = -E y_{-1} = \frac{\dot{p}_1}{X} \frac{\partial y_{-1}}{\partial \phi} \tag{6.4}
\]

and \( \partial p_o^* / \partial s = 0 \). This implies that \( \partial \xi_o / \partial \theta = p_o^* = 0 \).

The next order in \( \varepsilon \) yields

\[
\frac{\partial p_1^*}{\partial s} = (D \sigma B_1 + a) X_0 + (D \sigma K_1 + b) (Dy)_0 + A \xi_o + (D \sigma B_1 + s_1) \sigma_0 \tag{6.5}
\]

Averaging yields

\[
\frac{\partial \langle p_1^* \rangle}{\partial s} = \langle aX_o \rangle + \langle b(Dy)_o \rangle + \langle A \xi_o \rangle + \langle s_1 \sigma_o \rangle \tag{6.6}
\]
Here one has

\[
(Dy)_o = \dot{x} \left( \frac{\partial y_0}{\partial \sigma} + qs \frac{\partial y_{-1}}{\partial \phi} \right)
\]

(6.7)

so that

\[
\dot{x}qs \frac{\partial y_{-1}}{\partial \phi} = \langle (Dy)_o \rangle .
\]

(6.8)

As "basic" quantities we consider \( \xi_o \) and \( \langle p^*_1 \rangle \), in terms of which \( x_o \), \( (Dy)_o \) and \( \sigma_o \) have to be determined. \( \frac{\partial y_{-1}}{\partial \phi} \) then follows from equ. (6.8). To \( 0(\varepsilon^0) \) the equs. (4.12), (4.13) are

\[
(D_o \sigma_o \vec{B} \cdot \vec{B} + a_1) X_o + (D_o \sigma_o \vec{B} \cdot \vec{E} + b_1) (Dy)_o =
\]

\[
= (D_o a + c_2) \xi_o + (-D_o \vec{B} \cdot \vec{B} + s_2) \sigma_o ,
\]

(6.9)

\[
(D_o \sigma_o \vec{B} \cdot \vec{E} + a_2) X_o + (D_o \sigma_o \vec{E} \cdot \vec{E} + b_2) (Dy)_o =
\]

\[
= (D_o b + c_4) \xi_o + (-D_o \vec{B} \cdot \vec{E} + s_3) \sigma_o .
\]

In order to determine \( \sigma_o \) equ. (6.1) has to be solved to \( 0(\varepsilon^0) \). With

\[
\langle C_{-1}^+ \rangle = \langle C_{-1}^+ \rangle_t = 0
\]

one gets

\[
\gamma_{ou} = \int_{\theta} d\theta' \langle C_{-1}^+ \rangle_t - \frac{1}{\langle \lambda_o \rangle} \left( \langle \lambda_o \int d\theta' \langle C_{-1}^+ \rangle_t \rangle - \langle C_{-1}^+ \rangle \right) ,
\]

\[
\gamma_{ot} = \int_{\theta} d\theta' \langle C_{-1}^+ \rangle_t - \frac{1}{\langle \lambda_1 \rangle} \left( \langle \lambda_1 \int d\theta' \langle C_{-1}^+ \rangle_t \rangle - \langle C_{-1}^+ \rangle \right) ,
\]

(6.10)

where \( \lambda_q = 2\pi n q s \). The first terms do not contribute to density and pressure moments, respectively, since they are antisymmetric in sign \( v \).

When this is inserted into the quasineutrality condition (4.30) and the equations (4.31) for \( p_u, p_\perp \), the following coupled systems of integral equations for \( \phi_o, p_{uo} \) is obtained, together with an equation for \( p_{uo} \):
\[
\begin{align*}
\phi_0 & = \int_0^\infty \frac{\epsilon}{B} \left( \frac{\epsilon}{m} \right)^2 \frac{\partial F^\pm}{\partial \epsilon} = \int_0^\infty \frac{\Omega}{\epsilon} \left( \frac{\epsilon}{m} \right)^2 \frac{\partial F^\pm}{\partial \epsilon} \\
\cdot \{ < \frac{B}{v} \psi^-_o > + 2qs \frac{\partial}{\partial \theta} \left( \frac{1}{B} \int \theta < \frac{B}{v} \psi^-_o > \frac{B'}{v} \psi^-_1 > - < \int \theta < \frac{B}{v} \psi^-_o > \} \\
& + \int_0^\infty \frac{\epsilon}{\epsilon / B_{\text{max}}} \left( \frac{\epsilon}{m} \right)^2 \frac{\partial F^\pm}{\partial \epsilon} \left( \frac{\epsilon}{m} \right)^2 \frac{\partial F^\pm}{\partial \epsilon} \\
& \cdot \left[ \frac{\epsilon}{\epsilon / B_{\text{max}}} \right]^{t \to -} + \left[ \frac{\epsilon}{\epsilon / B_{\text{max}}} \right]^{t \to +} \left\{ \cdots \right\} \\
& - 2 \int \frac{\epsilon}{\epsilon / B_{\text{max}}} \left( \frac{\epsilon}{m} \right)^2 \frac{\partial F^\pm}{\partial \epsilon} \left( \frac{\epsilon}{m} \right)^2 \frac{\partial F^\pm}{\partial \epsilon} \\
& \cdot \left[ \frac{\epsilon}{\epsilon / B_{\text{max}}} \right]^{t \to -} + \left[ \frac{\epsilon}{\epsilon / B_{\text{max}}} \right]^{t \to +} \left\{ \cdots \right\},
\end{align*}
\]

where the brackets \{\cdots\} in the second equation are identical with those in the first one, and

\[
\Omega = \frac{\omega^2 \left( \frac{B}{v} \right)^2}{\omega^2 \left( \frac{B}{v} \right)^2 - (2\pi \hbar \chi q s)^2}.
\]

\(\psi^-_1\) and \(\psi_0^\pm\) are defined in equs. (4.23 to 4.26) as functions of \(y^-_1\) and \(\xi_0, y^\pm_0, \phi_0, P_{\perp 0}\), respectively.

The integral equations only depend on \(\omega^2\), as do the macroscopic equations (4.8) to (4.13). For \(w\) real and \(\dot{q} \neq 0\), \(\Omega\) may diverge, which
corresponds to the wave-particle resonance at \((i\omega - v \frac{\hat{B} \cdot \nabla}{B}) \tilde{f}^\pm = 0\). For unstable modes, \(w^2 < 0\), on the other hand, \(\Omega\) is finite.

The existence and properties of solutions of the integral equations are discussed in Appendix A. It is obvious, however, that, in general, no explicit solution may be given. This is also valid for \(w^2 \to 0\) since the trapped-particle term is independent of \(w^2\). Furthermore, even if \(\sigma_o\) were a given function no explicit solution of the coupled first-order system of differential equations (6.9) for \(X_o, (Dy)_o\) is available in general.

There is an additional complication which will become apparent in the next section: The \(O(e^1)\) equations which have to be taken into account introduce coupling to \(\sigma_1, y_1, \text{etc.}\) It is far from obvious how the coupling (ultimately between all orders) may be truncated.

One is thus forced to the conclusion that a further analytic discussion of stability with respect to the localized modes is not possible in general. There is one particular case, however, for which an explicit stability criterion can be derived. This case will be investigated in the next section.
7. Stability criterion for special $\sigma = \text{const}$ equilibria

Let us consider plasmas in which the number of trapped particles (at the resonant surface) may be neglected. In order to obtain criteria for marginal stability, we consider $0 > \omega^2 \to 0$. With $\Omega \to 0$ the solution to equ. (6.11), (6.12) becomes trivial:

$$P_{\perp 0} = -\frac{1}{\Gamma} \xi_0 \frac{\partial}{\partial r} P_{\perp 1} = -\frac{1}{\Gamma} \xi_0 \cdot \nabla P_{\perp 1},$$

$$P_{\perp 0} = \Gamma P_{\perp 0}, \quad \phi_0 = 0,$$

(7.1)

with $\Gamma = 1 + \frac{1}{B} \left( \frac{dP_{\perp 1}}{d\theta} \right)$. ($\Gamma > 0$ is required to avoid mirror instability (KADISH, 1966).) With equ. (4.15), (5.7) $\sigma_0$ may be expressed as

$$\sigma_0 = -\frac{P_{\perp 0}}{B} \frac{d\sigma}{d\theta}.$$

(7.2)

The differential equations (6.9) for $X_0$, $(Dy)_0$ may be explicitly integrated in the case $a_1 = a_2 = b_1 = b_2 = 0$. According to equ. (4.14) this requires

$$\sigma = \text{const}$$

(7.3)

(provided $\dot{B} \cdot \nabla B \neq 0$, $\dot{\xi} \cdot \nabla B \neq 0$), i.e. if the plasma anisotropy is constant on the rational surface. This will be assumed in the following. Equation (7.3) implies that

$$P_\parallel = P_1(r^1) - \frac{c}{2} B^2, \quad P_\perp = P_1(r^1) + \frac{c}{2} B^2,$$

$$\sigma = \Gamma = 1 + c,$$

(7.4)

where $c$ is an arbitrary constant, provided $c > -1$, $|c| \leq 2P_1/B^2$, in order to satisfy $\sigma > 0$, $P_{\perp 1} \geq 0$. Equation (7.3) also implies that
\[ c_1 = 0, \ i = 1, \ldots, 5, \text{ and } \sigma_0 = 0 \text{ even although the equilibrium is allowed to be anisotropic.} \]

One finally obtains

\[ \sigma \hat{B} \cdot \hat{B} \cdot \hat{X}_0 + \sigma \hat{B} \cdot \hat{K} \cdot (Dy)_0 - a \xi_o = C_1 , \]

\[ \tau \hat{B} \cdot \hat{K} \cdot \hat{X}_0 + \tau \hat{K} \cdot \hat{K} \cdot (Dy)_0 - b \xi_o = C_2 , \]

where the constants \( C_1, C_2 \) have to be determined from the next order in \( \varepsilon \). From equs. (4.12), (4.13) one obtains to \( O(\varepsilon^1) \)

\[ \dot{\chi} q s \frac{\partial}{\partial \phi} <\sigma \hat{B} \cdot \hat{B} \cdot \hat{X}_0 + \sigma \hat{B} \cdot \hat{K} \cdot (Dy)_0 - a \xi_o> = <s_2 \sigma_1> , \]

\[ \dot{\chi} q s \frac{\partial}{\partial \phi} <\sigma \hat{B} \cdot \hat{K} \cdot \hat{X}_0 + \sigma \hat{K} \cdot \hat{K} \cdot (Dy)_0 - b \xi_o> = <s_3 \sigma_1> + \]

\[ - \frac{\dot{p}_1}{\chi} \frac{\partial}{\partial \phi} <p_1^*> , \]

where most \( O(\varepsilon^1) \) quantities have disappeared by the averaging, but not \( \sigma_1 \). In Appendix B, however, it is shown that \( \sigma_1 \) is up-down symmetric, and since \( s_2 \), and \( s_3 \) are antisymmetric, the terms \( <s_2 \sigma_1> \), \( <s_3 \sigma_1> \) vanish. (Note that in the more general case, \( \sigma = \sigma(B) \), more \( O(\varepsilon^1) \) quantities such as \( <a_1 \hat{X}_1> \) etc. would remain in the equations.)

Comparison of equs. (7.6) with equs. (7.5), averaged over \( \theta \), determines

\[ C_1 = 0, \quad C_2 = - \frac{\dot{p}_1}{\dot{\chi} q s} <p_1^*> . \]

The determinant \( d \) of equs. (7.5) is

\[ d = \hat{B} \cdot \hat{B} \cdot \hat{K} \cdot \hat{K} - (\hat{B} \cdot \hat{K})^2 = [\hat{K} \times \hat{B}]^2 = (\nabla p_1)^2 . \]
When the solutions $X_0$, $(Dy)_0$ are inserted into equs. (6.8), (6.6), (6.4) one finally obtains

\[
\begin{align*}
 s^2 \frac{d\xi_0}{ds} &= c_{11} s \xi_0 + c_{12} <p_1^* >, \\
 s \frac{dp_1^*}{ds} &= c_{21} s \xi_0 + c_{22} <p_1^* >,
\end{align*}
\]

i.e. \( s^2 \frac{d\xi_0}{ds}^2 + 2s \frac{d\xi_0}{ds} + M \xi_0 = 0 \) with \( M = c_{11} c_{22} - c_{12} c_{21} - c_{11} \).

Here

\[
\begin{align*}
 c_{11} &= \lambda <\hat{B} \cdot \hat{Q}> , \quad c_{12} = -\lambda^2 <\hat{B} \cdot \hat{B} > , \\
 c_{21} &= <\hat{Q} \cdot \hat{Q}> + <A> , \quad c_{22} = -c_{11}
\end{align*}
\]  

and

\[
\begin{align*}
 \lambda &= -\frac{\dot{P}_l}{\dot{Q}^2} , \quad \hat{Q} = a \hat{K} + b \hat{B} , \\
 <A> &= -\sigma g_{mn} \hat{B}^m \hat{B}^n + \sigma \hat{B}^m \left( \frac{\partial B^m}{\partial r^1} - \frac{\partial B^m}{\partial r^m} \right) .
\end{align*}
\]  

The ansatz \( \xi_0 \sim s^v \) yields \( v = \frac{1}{2} (-1 \pm \sqrt{1-4M}) \). The condition

\[
1 - 4M > 0
\]

ensures that the solution is nonoscillating and serves as a necessary criterion for stability. (See PAO (1974) for a discussion of the matching procedure across the singular layer.)

It is straightforward to express the covariant and contravariant vector components in the criterion by means of quantities involving
\[ \hat{\mathbf{B}}^2, \mathbf{B} \cdot \mathbf{j}, (\hat{\kappa} = \sigma \hat{\mathbf{j}}) \], and covariant field line curvature \( \kappa_1 \). As a result, the stability condition is

\[
\frac{\chi^4}{4 \langle S \rangle} \left( \hat{q} + \frac{2}{\chi} \mathbf{R} \cdot \mathbf{S} \right)^2 - 2 \kappa_1 \frac{\mathbf{P}_1}{\sigma} - \hat{q} \chi^2 \langle \mathbf{R} \rangle - \langle \mathbf{R}^2 \mathbf{S} \rangle > 0 , \tag{7.13}
\]

where \( \mathbf{R} = \mathbf{j} \cdot \mathbf{B} / \mathbf{B}^2 \) and \( \mathbf{S} = \mathbf{B}^2 / (\mathbf{v} \mathbf{v})^2 \). This is identical with MERCIER's criterion, except that the hydrodynamic pressure \( P(V) \) is replaced by an effective pressure \( P_o(V) \),

\[
P_o = \frac{1}{\sigma} P_1(V) , \tag{7.14}
\]

where \( P_1 \) was defined in equs. (5.7), (7.4).

A posteriori, this simple result is not surprising: For \( \sigma = \text{const} \) the anisotropic equilibria are identical to MHD equilibria with the same magnetic field configuration but with \( P \) replaced by \( P_o \). In the linearized equations the plasma motion behaves as if it were isotropic, \( \sigma_{-1} = \sigma_o = 0 \), with no influence of \( \sigma_1 \neq 0 \), \( p_{n,O} \neq p_{\perp,O} \). In spite of the fact that, in general, \( \text{div} \xi_{-1} \neq 0 \), unlike in MHD, its actual value does not matter either, because only the combination \( X = \mathbf{D} \mathbf{x} - \text{div} \xi \) enters into the derivation of the criterion.

From equ. (7.4) one can deduce which direction of plasma anisotropy is favourable. Let us compare plasmas which are marginal with respect to the criterion (7.13), i.e. have the same \( P_o \) marg (and the same field configuration). The plasmas, however, may differ in their \( p_{n}/p_{\perp} \) ratio.

If we introduce \( \beta = 2(p_n + 2p_{\perp})/(3B^2) \) as a figure of merit, then

\[
\beta = \frac{2P_o}{B^2} \left( 1 + c + c \frac{B^2}{6P_o} \right) . \tag{7.15}
\]
This shows that among all plasmas with the same $P_0$, those with $c > 0$, i.e. $P_\parallel < P_\perp$, are better, within the framework of the present theory.
8. Conclusions

The guiding centre eigenmode equations for toroidal plasmas are investigated. Eigenmodes which are localized around a mode-rational surface and are almost constant along $\hat{B}$ (MERCIER-type modes) are shown not to exist for general anisotropic equilibria with $\sigma = 1 - (P_n - P_\perp)/B^2 = \sigma(V,B)$ ($V$ is the volume inside a flux surface). Such eigenmodes do exist for $\sigma = \sigma(B)$. No closed stability condition is obtained, however, because the mathematical problems involved do not have explicit solutions and because no obvious truncation procedure is available for the hierarchy of equations which result from expansion in the localization parameter $\varepsilon \ll 1$.

In the rather special case when $\sigma = \text{const}$ in the localization region and when trapped particles may be neglected the difficulties may be overcome and a necessary stability condition for axisymmetric equilibria is obtained. It is identical to MERCIER's criterion, except that the scalar pressure is replaced by an effective pressure $P_0(V) = (P_n + P_\perp)/2\sigma$. With the above-mentioned restrictions this implies, for example, that marginally MERCIER stable MHD equilibria can be loaded with higher plasma-$\beta$ if $P_\perp > P_n$ and still be marginal according to guiding centre theory.

The results obtained here do not exclude the existence of MERCIER-type stability criteria for general $\sigma(V,B)$ equilibria. Properly tailored test modes, when applied to guiding centre energy principles, could yield the desired result. These modes would not, however, be eigenmodes of the system.
Appendix A

The integral equations (6.11) for \( \phi_o, p_{\perp o} \) are of the form

\[
\phi(\theta) = \lambda \int d\theta' K(\theta, \theta')\phi(\theta') + r(\theta) . \tag{A1}
\]

with \( \lambda = 1 \) formally. Inspection shows that \( K(\theta, \theta') \) diverges logarithmically at \( \theta = \theta' \) but, with \( w^2 < 0 \)

\[
M^2 = \int d\theta d\theta' |K(\theta, \theta')|^2 < \infty \tag{A2}
\]

is finite. For \( \phi_o \) and \( p_{\perp o} \) \( M \) is \( O(1) \) and \( O(\beta = \frac{P}{B^2}) \), respectively.

Since for all eigenvalues \( \lambda_i \) one has

\[
|\lambda_i| \geq \frac{1}{M} \tag{A3}
\]

(MICHLIN, 1962), it follows that at least for \( \beta \ll 1 \) the equation for \( p_{\perp o} \) always has a unique solution, which may be obtained, for instance, recursively as a NEUMANN series.

The equation for \( \phi_o \), however, has an eigenvalue \( \lambda_{10} = 1 \) with eigenfunction \( \phi_{oo}(\theta) = \text{const}, \) at \( s = 0 \). Since, if (A2) is satisfied, eigenvalues do not accumulate, there is always a finite region around, and excluding, \( s = 0 \) where the equation for \( \phi_o \) also has a unique solution. At \( s = 0 \) existence of an inhomogeneous solution requires that the inhomogeneous term be orthogonal to \( \phi_{oo} \). This yields a side condition on \( y_o(\theta, s = 0) \) of the form

\[
\int d\theta c_1(\theta) y_o(\theta) = \xi_o , \tag{A4}
\]

where \( c_1(\theta) \) is antisymmetric.
Appendix B

Equations (7.5) show that $X_o$, $(\text{Dy})o$ are up-down symmetric. According to equ. (6.5) $p^*_1$ is also symmetric, provided that $B_1$, $K_1$ are antisymmetric. The latter may be proved by, for example, using the explicit representation of HAMADA coordinates, available for axisymmetric equilibria (LORTZ and NÜHRENBERG, 1974). Therefore $p^*_1$ does not contribute to $\sigma_1$ in $\langle s_{2,3}\sigma_1 \rangle$. The same holds for $\xi_1$, which according to equ. (7.2), disappears from $\sigma_1$ if $\sigma = \text{const}$. (The adiabatic $\xi^+_1$ terms behave identically in all orders.)

For free particles the non-adiabatic term of $O(\varepsilon^1)$ is governed by

$$g^+_1 = -\int \frac{d\theta}{o} \lambda^+_o g^+_{o} + \int \frac{d\theta'}{o} G^+_{o} + c_1$$

(B1)

where $c_1$ is constant and $\lambda^+_o$, $g^+_{o}$, $G^+_{o}$ are defined in equs. (6.1), (6.10). It follows that for $o^2 \rightarrow 0$ terms which contribute to even moments of $g^+_1$ are symmetric. This completes the proof that for the conditions of Section 7 $\langle s_{2,3}\sigma_1 \rangle = 0$ is valid.

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