PREFACE

Four years have passed since the first European Conference on Computational Physics in April 1972 in Geneva, an unintentionally long interval. In the meantime it has become apparent that a meeting on computational physics would be of particular interest if restricted to some fields of physics where similar numerical models are used and similar numerical problems arise. For this reason the second conference is concentrating on plasma physics and astrophysics. Both have to do primarily with plasmas under widely different conditions, strongly collision dominated or nearly collisionless, low temperature or relativistic. In addition, a cluster of stars formally behaves very similarly to an ensemble of charged particles.

A further reason for choosing the present fields is that, because of the complexity of real systems, numerical computations play a central role and hence have been developed to a high level of sophistication. It is hoped that this conference will demonstrate the close similarities between numerical plasma physics and astrophysics and provide an opportunity for researchers in one field to profit from advances in the other.

The present proceedings contain the extended abstracts of the conference contributions that have arrived prior to April 10. The review talks will be published separately in a special issue of Computer Physics Communications which should appear late in 1976. Since no parallel sessions are scheduled, a poster session was included in the conference programme for presentation of those papers that could not be incorporated in the main sessions for lack.
of time. This also provides an opportunity for post-dead line papers.

It is a pleasure to acknowledge the kind hospitality of Max-Planck-Institut für Plasmaphysik in hosting the conference and also the support given to the conference by Digital Equipment Corporation International as well as the generosity of Culham Laboratory for printing the conference poster. Finally, I should like to thank my colleagues in the Organizing and Programme Committees, and particularly R. Weinfurtner, for their help in preparing the Conference.

Garching, April 1976

D. Biskamp
The Conference is organized by the Computational Physics Group of the European Physical Society in collaboration with the Divisions of Plasma Physics and Astrophysics.

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*Pending confirmation*
Evolution of stars in the mass range 0.6 to 1.5 $M_\odot$

When stars finish burning hydrogen to helium in the core, the burning region moves outwards to form a shell source. Matter moves inwards through the burning region into the core which becomes progressively hotter and denser.

As the density of the core increases, the electron gas becomes degenerate. The hydrogen outside the hydrogen-burning shell forms a diffuse convective envelope of low density and great spatial extent (radius $\sim 5 \times 10^8$). The star is a red giant. Figure 1 shows the structure just before the helium flash.

The Site of the Helium Flash

In the core thermal conduction by degenerate electrons is efficient and there are no heat sources before helium burning starts, so to a first approximation the core is isothermal. However, heat is lost from the densest part of the core at the centre by the plasmon neutrino process

$$\gamma_{\text{plasmon}} \rightarrow \gamma + \gamma$$

A slight temperature inversion therefore appears, which causes helium burning to ignite some distance ($\sim 0.1 M_\odot$) from the centre. The peak temperature $T_\text{m} \sim 10^{9.5}$ K.

The rate of the reaction

$$^3\text{He} + ^4\text{He} \rightarrow ^7\text{Be} + \gamma$$

is proportional to about the 40th power of the temperature. Since the degenerate electron gas can take up much heat without a large pressure increase, a thermal runaway can occur.

The calculation of evolution up to the helium flash in this work was done with Eggleton’s code using mod-rewined physics. (Eggleton 1971 and 1972)

Helium Flash Computation assuming hydrostatic equilibrium

The energy generation rate in the helium-burning shell is great enough to require a convective zone outside it to transport the heat away. The temperature gradient is thus the adiabatic temperature gradient. The peak temperature rises and the convective zone extends; the rest of the star adjusts its structure adiabatically. When the electron degeneracy is lifted in the burning region expansion takes over and the peak temperature falls.

Computation of the evolution of the core through the helium flash is difficult with ordinary stellar evolution codes because the Henyey method normally used is unstable when the timestep becomes less than the thermal diffusion time. The Henyey method can be adapted for fast evolution, but a different approach has been used here.

We rely on the fact that once convection has started outside the helium-burning shell, the evolutionary timescale is much shorter than the thermal conduction timescale in the core. Thus changes in the internal energy occur only in the convective region; and here, because of the assumption of adiabatic convection, the temperature gradient is determined.

We therefore construct a sequence of equilibrium models each characterised by the value of the peak temperature (or, equivalently, the mass of helium burned to carbon). To obtain one model from its predecessor, first a new value of the peak temperature is chosen and the temperatures in the convective zone adjusted to keep the temperature gradient adiabatic; then the core model is adjusted to restore hydrostatic equilibrium by allowing adiabatic expansion of each zone. A suitable outer boundary condition is applied inside the hydrogen-burning shell, that is to say uniform expansion

$$\left( \frac{\rho}{\rho_0} \frac{d\rho}{dT} \right) T = 0$$

which has been demonstrated by earlier calculations (Thomas 1967, Zimmermann 1970).

Test calculations showed that this scheme gave results that were very nearly independent of the "time step" chosen, and that energy was conserved to within a few percent of the heat released by the $\alpha$ reaction.

Relationships between important timescales

In order to investigate the hydrodynamic behaviour, it is necessary carefully to consider the timescales appropriate to the various processes.

We define a nuclear timescale, $\tau_\text{n} = \frac{T}{\rho}$ for material burning at constant density in the absence of any heat flow. For helium burning it is inversely proportional to the square of the helium density and has a very strongly decreasing dependence on the temperature.

The convective timescale, $\tau_\text{c} = \frac{L}{U_\text{c}}$, where $L$ is the mixing length and $U_\text{c}$ the average speed in the turbulent convective motions. Since convection must distribute all the burning heat through the convective region, this timescale is related to the nuclear timescale.

To start with, $\tau_\text{n} \ll \tau_\text{c}$. If, however, the temperature becomes high enough so that $\tau_\text{n} > \tau_\text{c}$ then convection cannot take away the heat from the burning zone. We then have a burning zone of thickness roughly $\frac{T}{\tau_\text{c}}$ where the reaction rate is proportional to $T$, thermally isolated from the rest of the star. The burning zone is then moderated by convective cooling, and higher temperatures will be attained in the subsequent evolution.

If the temperature rises enough, the burning timescale may become less than the dynamic timescale, $\tau_\text{d}$, which is the time required for sound to travel across the burning region. This is less than 1 second. When this happens the assumption of hydrostatic equilibrium is no longer valid.
The present calculations

A small number of calculations through the helium flash have been made by other workers, each using somewhat different physical assumptions and treating a particular case (Thomas 1967 and 1970, Demarque and Mengel 1971, Zimmermann 1970). The indications are that the character and violence of the flash depend rather strongly upon some details of the physics which are not yet sufficiently certain. These include neutrino cooling rates, electron conductivities, the neutrino rate and its screening factor. The closer to the centre is the site of the flash, the more degenerate are the electrons, and the more violent is the event.

Calculations of the evolution through the flash will be presented for various values of appropriate parameters such as mass and composition of the star and the site of the flash. A typical quasistatic calculation for a core is shown in figure 2.

The effects of time-dependent convection and dynamics will be described as appropriate. This part of the calculation uses a Lagrangian mesh and an explicit difference scheme for the hydrodynamics.

References


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**Figure 1**
Temperature and density profiles for star of 0.75Mₜ and zero-age composition
X = 0.7, Z = 0.0004 at start of helium flash

**Figure 2**
Temperature - density plots for core of star of figure 1 during helium flash

**Model of Figure 1**
Computing Nonlinear Periodic Motions
of Stellar Envelopes and their Stability
by Kurt v. Sengbusch

Basis of the determination of radially symmetric motions
of stellar envelopes considered here, are the equations of
stellar structure, namely the equation of motion
\[ \frac{2u}{3t} + \frac{G M_r}{r^2} + 4r^2 \frac{\partial}{\partial r} \frac{2}{3} \frac{\partial^2 P}{\partial M_r^2} = 0 \]
the equation of energy
\[ \frac{2E}{3t} + \frac{2V}{3t} + \frac{3}{3} \frac{dE}{dM_r} = 0 \]
the equation of continuity in the integrated form
\[ \frac{dM_r}{3} = \frac{3}{3} \frac{dM_r^3}{dM_r} = V \]
and the definition of the velocity of matter
\[ u = \frac{3}{3} \frac{dM_r}{dM_r} = 0 \]
Internal energy E, specific volume V, radius r, and velocity u
are considered as dependent variables of the Lagrange Coordinates
M_r and t (M_r is the matter contained in a sphere of radius r). In
addition an equation of state P(E,V) and a transport equation
for L_e (the energy flux through a spherical surface of radius r)
are necessary to determine the other implicit variables P(E,V)
and L_e(E,V).

The inner boundary of the envelope is a stellar core of typically
90 - 95% of the star's mass contained in 10 - 20% of the radius,
which is at rest and provides a constant luminosity L. At the
surface of the star or outer boundary of the envelope the pressure P
will be zero and the temperature and luminosity will be connected
by the radiation properties of a grey atmosphere L_e = 8 \pi \sigma r^2 3 \frac{d}{d} t^4

In numerical calculations \(\tau = 2/3\) will usually be approximated
by \(\tau = 0\) because a separate calculation of the optical depth
is difficult to build into the scheme.

There are a number of types of variable stars for which periodic
light changes with a constant finite amplitude are observed
which are contributed to the periodic motions of the stars
envelope (δ-Cephei, RR-Lyrae, Mira-stars, δ-Sculi). A linear
perturbation analysis for equilibrium models of these stars
brought forward the understanding that certain transport
properties or energy generating mechanisms of the stellar matter
can be the source of self-excited motions of the stars matter.

Whereas a linear analysis will give periods and excitation rates,
a nonlinear analysis might explain certain features of the in
general by no means sinusoidal variations in luminosity and
radius of the star, in particular their finite amplitude of the motion.

The basis idea of this approach is, that the periodic
behaviour of a pulsating stellar envelope might be described by
a set of, say K, stellar models, adequately distributed over the period Π.
The structure of each model is given in terms of I (here r, u, E
and V) structure variables at J grid points in space \(M_{r,j}\). The
difference equations derived from the equations of stellar struc-
ture (first order in time and implicitly second order in space)
take the form
\[ \begin{align*}
& (5) \quad \sum_{K=1}^{K} \left( \sum_{I=1}^{I} \sum_{J=1}^{J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \sum_{I,J}^{I,J} \right)\ \Pi_{i,j} = 0, \quad j = 1, \ldots J,
& \quad k = 1, \ldots K
\end{align*} \]
Here we have already introduced the dimensionless time
\(\tau = t/\Pi\) and assumed, that \(\tau_k\) will not be changed during the
iteration for a solution.

Periodicity of the wanted solution requires \(Z_{i,j} - Z_{i,j}^{K} = 0\)
(i = 1,...I; j = 1,...J) and to determine a unique solution
of the set which are just phase-shifted against each other an
additional requirement, for instance, of the form \(Z_{i,j}^{K} = 0\)
is needed. So we are left with last one plus nonlinear difference
equations for the unknowns \(Z_{i,j}^{K}\), which we are going to solve
with the Newton-Raphson Method. With \(Z_{i,j}^{K}\) being a trial set of
the unknown quantities, we compute corrections \(\delta Z_{i,j}^{K}\), \(\delta \Pi\) from
solving
\[ (6) \quad \delta G(Z_{i,j}^{K}, \Pi) = - G(Z_{i,j}^{K}, \Pi) \]
According to the special form of the difference equations (6)
allows the representation
\[ A_k Z_{i,j}^{K-1} + B_k Z_{i,j}^{K} + C_k \delta \Pi + d_k = 0 \]
with the elements of \(A_k\) being \(a_{i,j}^{K-1}\), of \(B_k\) being \(a_{i,j}^{K}\),
\(c_k\) being \(a_{i,j}^{K}\) and \(d_k\) being the residuals \(a_{i,j}^{K}\).
Similarly the notation \( z_k \) and \( \delta z_k \) will represent a vector of the ensemble \( z_{ij}^k \), resp. \( \delta z_{ij}^k \) (\( i=1..I, j=1..J \)) for a particular timestep \( k \). If we now write

\[
(7) \quad \delta z_k = A_k^* \delta z_{ij}^1 + c_k \delta \Pi + d_k
\]

the recursion formulae

\[
A_{k+1} = (-B_{k+1})^{-1} A_{k+1} \delta z_k
\]

\[
c_{k+1} = (-B_{k+1})^{-1} (A_{k+1} c_k + c_{k+1})
\]

\[
d_{k+1} = (-B_{k+1})^{-1} (A_{k+1} d_k + d_{k+1})
\]

will combined with the periodicity requirement lead to

\[
(9) \quad \delta z_{ij}^1 = (E - A_k)^{-1} \left\{ c_k \delta \Pi + d_k - (z_{ij}^1 - z_k) \right\} = c_\Pi + d
\]

The phase condition might now be used to determine \( \delta \Pi \) and thus \( \delta z_{ij}^1 \) is determined. With \( \delta z_k = (-B_k)^{-1} (A_k \delta z_{ij}^1 + c_k \delta \Pi + d_k) \)

the corrections for all \( k \) models can be determined.

(7) indicates for \( k = K \) how corrections at time \( t = \Pi \) are coupled to corrections at time \( t = 0 \). This allows us to determine the corrections \( \delta z^* \) to our initial model at \( t = 0 \) such that its evolution behaves periodically. This Newton-Raphson-Method will work in principle but it will in general not be easy to provide a sufficiently good approximation \( z, \Pi \) for the method to converge, or considerably small undercorrection factors have to be used in order that the corrected model still lies in the approximately linear neighbourhood of the approximation model. Different procedures may be used to speed up convergence by an intermediate nonlinear iteration step. We start off with a full set of approximate models which are a priori periodic and apply the full corrections only in some inner part of the envelope. We then integrate the outer part through some 2-3 periods with periodic inner boundary conditions as given by the outer boundary of the inner part of the envelope. As the difficulties which hinder strong convergence of the full scheme are concentrated very close to the surface this integration yields an almost periodic outer part of the envelope which fulfills all difference equations.

In many cases we could find more than one solution for a particular set of inner boundary conditions that is to say, a core radius and luminosity, namely fundamental mode and 1. overtone solutions.

The question comes up which of the solutions would persist in reality and could the answer to this question possibly be dependent on the history of the star that is to say the evolution of the stellar core.

Equation (7) for \( k = K \) indicates how a perturbation of the model at \( t = 0 \) evolves through one period, because for the periodic solution \( \delta z_k = 0 \) and for the present question \( \delta \Pi = 0 \) can be assumed. The eigenvalues of Matrix \( A_k^* \) will therefore determine stability of the periodic solution obtained. Eigenvalues \( \lambda \) with \( |\lambda| > 1 \) will indicate growth of the corresponding eigensolution \( \delta z_\lambda (0) \).

The form of the eigenfunction \( \delta z_\lambda \) and the period determined from the phase angle of \( \lambda \) may relate it to eigenfunctions of a linear stability analysis of the equilibrium model.

In that sense we have tested fundamental mode periodic solutions for the behavior of 1. overtone perturbations and vice versa. The result is shown in Fig. 1. The stability analysis can separate for the range of physical parameters in question the range \( T_e > 6300^\circ K \) for which only 1. overtone periodic solutions are stable and the range \( T_e < 6100^\circ K \) for which the fundamental mode pulsation is favoured. For \( 6100^\circ K < T_e < 6300^\circ K \) both solutions are stable and the actually observed mode will depend on the evolution of the star. An evolution with growing radius at constant luminosity, this means evolution with decreasing effective temperature \( T_e \), will favour first overtone pulsations in this range and vice versa.

![Fig. 1](image)

a) Linear \((\omega/\omega_R)\) and nonlinear \((E_{kin})\) results of stability analysis for a star of \( M = 0.7 M_\odot, L = 44.7 L_\odot, X = 0.7, Z = 0.0004 \)

b) Stability analysis of nonlinear periodic motion at full amplitude
The gravitational collapse of a matter-antimatter symmetric gas sphere

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The symmetry of the properties of matter and antimatter in laboratory physics has led to speculation on the possibility of matter and antimatter being equally abundant in the universe (1), (2). A particularly simple model of the metagalaxy (the observed universe) incorporating the matter-antimatter (M-AM) symmetry principle is that of Alfvén and Klein (3) and Alfvén and Bonnevier (4). In this model, the metagalaxy is a spherical cloud of M-AM plasma which initially has a sufficiently low density for proton-antiproton annihilation events to be negligible; however, the density increases as the metagalaxy collapses under its own gravitational field, and the annihilation rate consequently increases. The annihilation radiation is envisaged as reaching a sufficient intensity for radiation pressure to overcome the gravitational force and reverse the collapse to an expansion, thus leading to an expanding metagalaxy in accordance with observations. Alfvén and Bonnevier (4) have shown that such a reversal can occur using Newtonian gravity and a very simple model of radiation transfer; this paper reports the results of calculations of a similar system using General Relativistic gravity and an improved model for radiation transfer. Calculations of this type are of interest in considering the possibility of observing antimatter in the universe, regardless of cosmological model, since they furnish information on the luminosity, dynamics and lifetime of smaller scale M-AM gas spheres (of say a galactic mass) for comparison with possible candidates for such an object.

The spherically symmetric metric $g^{\mu\nu}$ used in the calculation is that of Misner and Sharp (3), viz.

$$g^{\mu\nu} dx_{\mu} dx_{\nu} = ds^{2} = \left(\frac{r^{2} dr}{1 + r^{2} \mu^{2} \nu} \right) dt^{2} + r^{2} \left(\frac{d\theta^{2}}{\sin^{2} \theta} + d\phi^{2} \right) - \frac{2}{\nu} dr + r^{2} \left( d\theta^{2} + \sin^{2} \theta d\phi^{2} \right) - \frac{2}{\nu} \partial_{t} t^{2}$$

(1)

where $(r, \theta, \phi)$ are spherical polars, $R, m$ and $\nu$ are functions of $r$ and $t$, and $U = e^{\nu} dr / dt$ is the scalar factor for the sphere and $U$ gives a measure of the velocity of collapse. The total energy-stress tensor $\hat{T}^{\mu\nu} = M^{\mu\nu} + E^{\mu\nu}$ where the M-AM and radiation energy-stress tensors are respectively given by

$$M^{\mu\nu} = \hat{\nu} \nu^\mu \nu^\nu$$

(2)

and

$$E^{\mu\nu} = \lambda_0 \alpha \nu^\mu \nu^\nu + \lambda_0 \nu^\mu \nu^\nu + \beta \nu^\mu \nu^\nu$$

(3)

$\lambda_0$ is the proper M-AM mass energy density, $\lambda_0$ is the 4-velocity of M-AM (c = 1, $\nu = \mu = 1$) and $\alpha$ and $\beta$ are the energy densities in the comoving frame of M-AM of an isotropic radiation field and a unidirectional radiation field respectively; $\beta$ consists of photons moving radially with 4-velocity $\nu^\mu$ in the M-AM comoving frame ($k^\mu = 0$, $k^\mu = -1$). We neglect M-AM pressure on the basis that if $\alpha, \beta, \lambda_0$ then pressure gradient forces will be negligible compared to radiation forces.

The energy stress tensors satisfy

$$M^{\mu\nu} \nu^\nu = - E^{\mu\nu} \nu^\nu = F^\mu$$

(4)

where $F^\mu$ is a 4-force representing transfer of momentum between M-AM and radiation by photon scattering, and transfer of energy by annihilation. The form of $F^\mu$ is

$$F^\mu = - \nu^\mu \nu^\nu \left( \alpha \nu^\mu + \lambda_0 \nu^\mu \nu^\nu + \beta \nu^\mu \nu^\nu \right)$$

(5)

where $\lambda_0 = \frac{\rho}{m}$ and $\nu^\mu \nu^\nu = \gamma$ and $\beta = \lambda_0 \nu^\mu \nu^\nu$ are adjustable parameters in the vicinity of unity.

The coordinate conditions employed are $u^1 = u^2 = u^3 = 0$ (comoving coordinates), and $y^4(t, r) = 0$ where $r$ is the radius at the boundary of the sphere. Finally, the boundary conditions on the energy stress tensors are $\hat{T}_{\nu^\mu}^{\nu^\nu} = 0$, $\alpha_{\nu^\mu}^{\nu^\nu} = 0$ and $\beta_{\nu^\mu}^{\nu^\nu} = 0$.

Substituting (1), (2), (3) and (5) into the Einstein Equations and the interaction equations (4), we obtain a set of first order partial differential equations in $r, t$ for the variables $R, \alpha, \beta, \gamma, \nu$ and $U$. These are non-linear but may be linearized by the choice of variable $\gamma$ and $U$ such as variables changing by using an implicit integration scheme with the triangular algorithm being necessary only for $\alpha$ and $\beta$.

The four diagrams show the $r$-dependence ($r = 1$) of $\gamma, \alpha, \beta$ and $U$ at four stages of a typical reversal solution. The initial M-AM mass for this run is $1.8 \times 10^{16}$ M$_{G}$ and $\gamma = \gamma = 1$. The energy densities $\alpha, \beta$ and $U$ are plotted logarithmically (left-hand scale) as fractions of $3 \times 10^{-4}$ ergs/cm$^3$, while the "velocity" $U$ is plotted linearly (right-hand scale) as a fraction of $c$. The initial $\gamma$ is $t = 0$ is uniform in most of the sphere but drops to zero at the boundary; the initial $\gamma$ is that of free-fall from $R = \infty$, where $\gamma$ is 0. Initially $\alpha$ is 0. $\gamma$ is initially 1. In free-fall $\gamma$ and $U$ would retain their initial shape with $\alpha$ and $\beta$ increasing, but we see that at $t = 0.248$ (time unit = 10$^{8}$ years) a pronounced shell has appeared in $\gamma$ at the boundary of the sphere. This is due to two consequences of the energy transfer by annihilation from $\gamma$ to $\alpha$ and $\beta$: (1) the slowing of the collapse by $\beta$ is more effective near the centre than on the boundary, and (2) $\beta$ transfers gravitating mass/energy from the centre towards the boundary. Both these effects cause the centre of the sphere to collapse less rapidly relative to the boundary region than it would do in free-fall, leading to the compression shell at the boundary. The consequent shell in $\alpha$ through annihilation in turn leads to a peak in $\beta$ on the outer side of the shell. The formation of the shell seems to be instrumental in effecting the reversal of the collapse, and already at $t = 0.248$ strong deceleration of $\alpha$ has begun at the peak in $\beta$. By $t = 0.250$ $\alpha$ has been driven positive near the boundary, and the divergent $U$ has now begun to break up the shell. A secondary peak in $\beta$ propagates towards the centre with the peak in $\alpha$ and $\beta$, the region of positive $U$ also propagates towards the centre. By $t = 0.262$ complete reversal of the collapse throughout the sphere has occurred, and the mass of M-AM left in the expanding sphere is $2.8 \times 10^{15}$ M$_{G}$. Above a certain critical initial M-AM mass reversal of the collapse does not occur, and for $\xi = 1, \gamma = 1$ we obtain a critical mass $\sim 10^{19}$ M$_{G}$ (unit of $R = 10^{8}$ light years).

REFERENCES

RELATIVISTIC GRAVITATIONAL COLLAPSE TOWARDS THE BLACK HOLE STATE

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In the 1960s a number of people carried out computer calculations of relativistic gravitational collapse of spherical, non-rotating bodies. The most extensive of these investigations was by May and White.

It is convenient to divide the collapsing matter into a system of concentric spheres and to use the rest mass (\(\mu\)) contained within a particular sphere as the co-moving radial coordinate of its surface. For spherical symmetry, the general form of the metric line element can be written as

\[
ds^2 = -A^2 dt^2 - 2ABdtd\mu + C^2 d\mu^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

where \(A, B, C\) and \(R\) are functions of \(\mu\) and \(t\). One is also free to set either \(B = 0\) or \(C = 0\) which correspond respectively to a choice of "cosmic" time or retarded "observer" time as the time coordinate. "Cosmic" time makes the metric diagonal, giving certain simplifications and this was chosen by the early investigators. An explicit two-step computer code is used for the calculations and the time-step is adjusted so as to satisfy the relativistic generalization of the classical Courant stability condition. Neutrino processes are not considered in these calculations.

May and White computed gravitational collapse of bodies starting from constant density. All of their results (for bodies of various masses and equations of state) show either a continued collapse, at the end of which a singularity forms inside the Schwarzschild radius, or a bounce, where the initial collapse is halted and the matter settles into an equilibrium configuration together with an expanding shell of ejected matter which carries away the binding energy. In late stages of continued collapse pressure forces no longer help to support the body but instead accelerate collapse and some mass zones then give negative contributions to the total energy. The singularity occurs first at an intermediate mass zone and not at the centre.

There are two main drawbacks in the calculations of May and White: firstly, that homogeneous initial conditions give rise to several non-physical effects; secondly, that in a continued collapse with "cosmic" time it is not possible to follow all of the observable motions of the outer layers. The computation has to be terminated as soon as a singularity forms. More recently I have produced results for collapse from non-homogeneous initial conditions, using both "cosmic" time and "observer" time. Fig.1 shows the behaviour of the "observed velocity" \(R_t\) (where \(t\) is the "observer" time) for a \(2M_\odot\) body with a \(Y = 5/3\) polytropic equation of state.

![Fig.1. The "observed velocity" \(R_t\) is plotted against \(\mu\) at various times during the collapse of a \(2M_\odot\) non-rotating body; \(t\) is the "observer" time.](image)

There are, as yet, no equivalent hydrodynamical computations for collapse of rotating bodies but much can be learned by studying sequences of equilibrium models. Chandrasekhar and I studied a sequence of slowly rotating
homogeneous configurations which was constructed so as to mimic an adiabatic collapse to infinite central redshift. The metric for a slowly rotating body in equilibrium can be written as

$$ds^2 = -e^{2\varphi}(1 + 2\varphi)dt^2 + e^{2\lambda}(1 + 2\lambda)dr^2 + r^2(1 + 2\kappa)[d\vartheta^2 + \sin^2\vartheta(d\varphi - \omega dt)^2]$$

where $h$, $m$, $k$ and $w$ are functions of $r$ and $\vartheta$ and are considered to be small perturbations on the standard spherical non-rotating metric. All equations are expanded in powers of $R$ (the angular velocity which is assumed to be independent of $r$ and $\vartheta$) and terms of order $\ll R^2$ retained. The sequence is constructed by choosing progressively increasing values for the central redshift and calculating the corresponding equilibrium models. The quantity $J/A^2$ is kept constant along the sequence ($J$ is the total angular momentum and $A$ is the gravitational mass). This quantity (the "shape factor") would, indeed, remain constant in the present approximation for a dynamical collapse which was adiabatic apart from the emission of gravitational radiation.

During collapse the ellipticity of the body rises to a maximum and then decreases again (see Fig.2). At high redshifts the levitational effect of rotation reverses and at infinite central redshift the external metric is the Kerr metric to the accuracy of the approximations.

The behaviour of the ellipticity is qualitatively different from that predicted by Newtonian theory which says that the body will become progressively more oblate during collapse and end up as a disc. The physical reason for the different behaviour in general relativity is as follows. The quantity which determines centrifugal effects is not $\varphi$ but $\varpi = \varphi - w$ which is the coordinate angular velocity of the matter relative to a local zero angular momentum observer. $\varpi/R$ decreases monotonically during the collapse and, despite the fact that $\varpi$ continues to increase, the centrifugal effects (and hence the ellipticity) reach a maximum and then decrease again.

Similar calculations for configurations with a density profile give results which are consistent with those for homogeneous models but the maximum central redshift at which the body is stable is lower than in the homogeneous case.

I am, at present, developing a computer code to calculate dynamical gravitational collapse of slowly rotating bodies. This code combines the methods of the two sets of work which I have been describing. The calculation again uses a Lagrangian grid but instead of dividing the matter assembly into a series of spherical shells one now considers a series of oblate isobaric surfaces which are again parametrised by the rest mass contained within them. The models start from rest with uniform rotation but subsequently the angular momentum of each fluid element is conserved giving rise to a differential rotation. The gravitational radiation can be calculated and changes of shape will be monitored as the configuration approaches the black hole state. The first results should be available within the next few months.

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A MAGNETOSTATIC PARTICLE CODE AND ITS APPLICATION TO STUDIES OF ANOMALOUS CURRENT PENETRATION OF A PLASMA

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A large number of important plasma problems involves self-consistent magnetic fields. For disturbances which propagate slowly compared to the velocity of light the magnetostatic approximation (Darwin model) suffices. We have developed a two and one half dimensional particle model based on the Darwin model to investigate such problems. A finite sized particle model is employed and the dipole approximation is used. The electromagnetic field is computed in Fourier space. The fields and current are decomposed into longitudinal and transverse components. The solutions for the longitudinal electric field and magnetic field are obtained from Poisson's equation and Ampere's law. The solution for the transverse electric field is more complex involving the time derivative of the magnetic field and through it the time derivative of the current which involves the transverse E field. We obtain an equation for the transverse E field which involves only the instantaneous E and B fields. The equation is as follows

\[\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\mathbf{B}_0}{c^2} \times \nabla \times \mathbf{E} = \frac{\mathbf{B}_0}{c^2} \left( \mathbf{E}_L + \mathbf{E}_T \right) + \frac{4\pi}{c^2} \left( \frac{n_1}{n_0} \mathbf{V}_i + \mathbf{V}_e \right) - q_1 \nabla (\nabla \cdot \mathbf{E}_T) \right) \cdot \mathbf{f} (\mathbf{r}, \mathbf{r}_1) \]

where \(\mathbf{f} (\mathbf{r}, \mathbf{r}_1)\) gives the charge distribution around \(\mathbf{r}_1\), the summation is for both electrons and ions,

\[u_{p_0}^2 = \frac{4\pi e^2 n_0}{n_e} \left( 1 + \frac{n_0}{n_i} \right) \mathbf{f} (\mathbf{r}, \mathbf{r}_1)\]

and

\[u_{p_0}^2 = \frac{4\pi e^2}{n_e} \left( n_e - n_0 \right) \cdot \frac{n_0}{n_i} (n_1 - n_0) \mathbf{f} (\mathbf{r}, \mathbf{r}_1)\]

In Fourier space, Equation (1) can be written in matrix form. If the density perturbation is small, a Jacobian iteration scheme can be used to yield a rapidly converging solution for \(\mathbf{E}_T(k)\). For large density perturbation (exceeding a few times \(N_e\)), one can subtract any unit of \(\omega_{pe}^2/c^2(E_T)\) from both sides of Eq(1) to make the off-diagonal elements smaller than the diagonal elements and proceed with the same iteration scheme. Using this procedure to find the transverse E field leads to a numerical procedure which is stable and conserves energy.

The code has been extensively tested and yields accurate results for various types of wave propagation in a thermal plasma. The results are shown in Fig. 1. With \(\mathbf{E} = 0\), alfin, ion cyclotron, and whistler waves are observed, whereas with \(\mathbf{E} = 0\) magnetosonic and electromagnetic upper hybrid waves are seen. Waves propagating at an oblique angle to the external magnetic field are also shown in Fig. 1. The simulation results fall closely on the theoretical dispersion curves. By employing vacuum boundary conditions we are able to investigate current penetration into the plasma. The time evolution of induced plasma current is shown in Fig. 2. At early time the current profile follows the classical skin formation; i.e. \(J_p(x) = J(0) e^{-x/L}\) which gives \(J(0)/J(L/2) = 4\). At later time \((\omega_{pe} t = 200)\) this ratio was reduced to 2.6. The anomalous resistivity will be evaluated. The power spectrum from simulation results shows that the lower hybrid is the dominant mode during the current penetration process.
Theory and Simulation on the Anomalous Transport
due to Drift Wave Instabilities

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With the advent of tokamak research, gradient-driven microinstabilities in a magnetically confined and MHD-stable plasma have attracted increasing interest in recent years. While much of the present theoretical work has been directed at obtaining relevant stability criteria for various microinstabilities, the understanding of their nonlinear behavior and the associated plasma transport is still at its infancy. It is generally believed that the particle code simulation should play a vital role in helping us to gain insight into these areas and provide guidelines for the analytic work.

With that in mind, we have conducted extensive numerical simulations to study the collisionless drift wave instabilities\(^1\) (universal instabilities) using a 2-1/2 dimensional electrostatic particle simulation code capable of handling a nonuniform and bounded system.\(^2\) Results for the nonlinear evaluation of the stability and the accompanied particle and energy transport will be reported along with the scaling laws in the presence of shear. We will also report the theoretical investigation on the universal model using a self-consistent quasilinear approach based on the observation of the simulation results.

The slab model used for the simulation is shown in Fig. 1. The plasma is nonuniform in \(x\) following a prescribed density profile and is bounded by two conducting walls where the electrostatic potentials are zero. In the \(y\) direction, the plasma is uniform and is assumed periodic so that the wave can propagate through the system. The main magnetic field \(B_0\) is perpendicular to the inhomogeneous \(x\) direction with \(B_0x > B_0y\). The sheared magnetic field \(B_s\) is in the \(y\) direction. The parameters for the simulation are:

\[\lambda_D = 2, \quad \omega_e/\omega_p = 2, \quad \langle N \rangle, \quad \text{(average number density)} = 8, \quad \text{a (particle size)} = 1.5 \quad \text{and} \quad \theta = 88.5^\circ.\]

Two density profiles have been used in the calculations; the exponential profile given by

\[n_0(x, t = 0) = \langle n \rangle \exp(-x/x_0),\]

and the hyperbolic tangent profile given by

\[n(x, t = 0) = \langle n \rangle \left[1 - \tanh(x - x_0)/\ell_0\right],\]

where \(\langle n \rangle = 24/n\) and \(x_0 = 32\).

For the instability driven by the finite Larmor radius effects, the system described above has essentially one unstable mode, \(i.e., \quad n = 1\) in the \(y\) direction, according to the linear theory.\(^3\) The measured growth rate and frequency of the instability have confirmed this prediction. The time evolution of the density profiles is shown in Fig. 2(a), and the corresponding mode structures for the perturbed potential \(\phi_1(x)\) is shown in Fig. 2(b). As we can see, the density profile undergoes a quasilinear change in time due to the \(E \times B\) drift as the result of the instability. Initially, the diffusion takes place at around the maximum wave amplitude, then gradually evolves into a plateau, and finally settles into a new stable configuration. The simulation results have also shown that in the linear stage of the instability there is energy transfer from the electron parallel temperature to the waves through inverse Landau damping in the unstable regions. However, as the waves grow in amplitude causing the quasilinear diffusion, a significant amount of the electron heating has been observed elsewhere. After the plasma reaches the stable configuration, the heating also stops. The electron heat transfer patterns are shown in Fig. 2(c). No ion heating has been detected in the simulation.

The results described above strongly imply that the mechanism of the nonlinear saturation is the quasilinear diffusion in the configuration space and the observed anomalous energy transport is the result of the Landau absorption of the wave energy due to the density profile change which alters the local dispersions.

With these observations, we have developed a self-consistent quasilinear model to describe the nonlinear evolutions of the instability. The analysis is based on the assumption that the time rate of change for the wave frequency \(\omega\), the density \(n\), and the temperature \(T\) is much smaller than those of the linearly perturbed quantities. Using the multiple-time-scale expansion for the linearized Vlasov equation, we obtain the normal mode equations for the eigenfrequency and the amplitude of the wave as follows,

\[
\begin{align*}
\left( \frac{d^2}{dx^2} + \sum_{\alpha} \left( \frac{\omega_{\alpha}}{\omega} - \frac{T_\alpha}{\omega} \right) C_{\alpha} / \omega \right) \phi_\alpha(x) &= 0, \\
\sum_{\alpha} C_{\alpha} / \omega \phi_\alpha &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{\tau_k} \sum_{\alpha} \frac{\omega_{\alpha}}{\omega} \left( \frac{T_\alpha}{\omega} - 1 \right) Z_{\alpha} &= \omega + \omega_k, \\
\sum_{\alpha} \frac{\omega_{\alpha}}{\omega} \phi_\alpha &= 0.
\end{align*}
\]
where

\[ C_a = X_{\alpha a} \left( \left[ \frac{1}{2} \frac{T_a^{1/2}}{E_a} + \frac{\omega^*_{\alpha a} (1 - \eta_{\alpha a})}{E_{\alpha a}^{1/2}} \right] Z_a - \frac{E_{\alpha a}^{1/2}}{2} \right), \]

Here \( \alpha \) denotes the species, \( X_{\alpha a} = \int \phi_a b_a \exp(- b_a) \), \( b_a = k_{\alpha a}^2 \), \( Z_a = 2 \epsilon_\omega \), \( \epsilon_\omega = \frac{\phi_a}{E_{\alpha a}^{1/2}} \), \( \eta_{\alpha a} = \frac{\partial n_a}{\partial n(n)} \), and \( \omega^*_{\alpha a} \) is the diamagnetic drift frequency.

The particle diffusion and the energy transport of the background plasma due to the instability can be calculated from the fluid equations

\[ \frac{3}{3t} n_a + v \cdot (n_a u_a) = 0, \quad (3) \]

\[ \frac{3}{3t} \left( n_a^{1/2} + n_{a}^{1/2} \right) + v \cdot (n_a u_a) = -\frac{q_a}{m_a} \int v^2 e_{\alpha a} \frac{3 v^3}{3 v} dv = 0, \quad (4) \]

where \( \overline{v} \) denotes the average of \( v \) over the fast time scale 2\( \pi / \omega \).

At every time step \( \Delta t \ll 2 \pi / \omega \), the eigenfrequency and the amplitude of the wave can be calculated from Eqs. (1) and (2) using the existing density and temperature profiles. These values of \( \omega \) and \( \phi_\omega \) can then be used in Eqs. (3) and (4) to update the background \( n \) and \( T \). The theoretically calculated results are in reasonable agreement with those from the simulation. Details will be presented.

With the introduction of the sheared magnetic field, we have observed the reduction in the anomalous plasma transport with the overall patterns similar to those of the shear-free cases. The measured particle diffusion and heat conductivity coefficients using the usual diffusion equation \( \frac{3}{3t} \left( n_a^{1/2} + n_{a}^{1/2} \right) + v \cdot (n_a u_a) = \frac{q_a}{m_a} \int v^2 e_{\alpha a} \frac{3 v^3}{3 v} dv \), is shown in Fig. 3, where \( \overline{\langle n_a \rangle} = 1/0.07 \) is the effective density scale length and \( \overline{\langle n_a \rangle} \) is the density scale length. For the shear-free case, \( \delta_0 \) is comparable to the values given by the Bohm diffusion and the turbulent diffusion of \( \gamma/k_0^2 \). On the other hand, \( \gamma_e \) is much larger than \( \delta_0 \) because of the Landau absorption of the wave energy. Figure 3 also shows that the heat conductivity of the electrons for the partially shear-stabilized drift waves is much less sensitive to shear than the particle diffusion.

We have also carried out simulations with the real mass ratio between the electrons and ions using a drift approximation for the electron transverse motion. The results obtained are similar to those reported above. Moreover, we have investigated the cases involving turbulent drift waves where many linearly unstable modes are present simultaneously in the system. Coupling between different \( k_0 \) modes has been observed. However, quasilinear density profile decay is again the dominant nonlinear saturation mechanism. Details will be presented. Simulations of the three-dimensional drift waves are now being carried out using a newly developed 3-d code which allows coupling between the modes along the \( B_0 \) direction. These results will also be presented.


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Numerical Simulation of Dissipative Trapped-Electron Instability

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Low-frequency microinstabilities such as drift wave and trapped particle instabilities have been of wide interest in relation to research on controlled thermonuclear fusion in low-density magnetic confinement systems. While there is certain experimental evidence that these low-frequency instabilities may be responsible for the anomalous plasma transport, positive identification of the instability and the scaling of the associated plasma transport are often difficult in many cases where the experimental situation is far more complex than the theoretical treatment.

We have developed a simulation model, shown in Fig. 1, to study the trapped electron instability in a linear geometry. The density and the temperature are nonuniform in the x direction where the plasma is confined between two conducting plates located at x = 0 and x = L. In the y and z directions, the plasma is assumed periodic. The mirroring field is imposed by the reflection of particles at the plane z = z_0 and z = z_1 which is assumed to be the same plane because of the periodicity of the model. Monte Carlo collisions are used to model the electron pitch angle scattering with the ions. A 2 1/2 dimensional model has been used for the actual calculations which requires a small but finite angle between B and z axis.

The dissipative trapped-electron instability in the original treatment of Kadomtsev and Pogutse is described by the dispersion relation

\[ 1 - \frac{\omega}{\omega^*} \frac{\dot{k}^2 e}{m_1 e^2} = \frac{1}{2} \int \frac{f_M d^3 \nu}{\omega - \omega^* + \nu^2 (\nu^2/c^2 - 3/2)} \frac{d^3 \nu}{\omega + \nu^3/c^3} = 0 \]

where \( f_M \) is the Maxwellian electron distribution function, \( \epsilon \) is the inverse aspect ratio, \( \omega^* \) is the electron collision frequency, \( \omega = -c_k T_e (dn/dx)/eBn \) and \( \eta_0 = d\eta_e e^2/dm_1 \). The above equation predicts an instability for \( \omega < \omega^* < \nu/e/c \) with growth rate

\[ \gamma = \epsilon^{3/4} \eta_0 \omega^*/\nu_0 \text{ for } \omega^*_b < \omega < \omega^*_c \]

where \( \omega_b, \omega_c \) are the bounce frequencies of the electrons and the ions.

Typical simulation parameters are the following: 64 x 64 mesh, 5^4 particles, \( m_1/e^2 = 100 - 400 \), \( T_e/m_1 = 4 - 16 \), \( \eta_0/e^2 = 2 \), and \( L_p/\Delta = 0.5 \) where \( \Delta \) is the mesh size. Initial density and electron temperature profiles are taken as

\[ N(x)/\bar{N}, T_e(x)/\bar{T}_e = [1 - \eta_{N,T} \tanh(x - L/2)/L_{N,T}] \]

Simulations were carried out for various values of \( \eta_0 \) and the collision frequencies.

Figures 2(a)(b) show the density modulation \( \delta n/n_0 \) and the electric field as the instability develops. Large scale perturbations \( \delta k_{e,1} = 0.25 \) and \( \delta k_{e,2} = 0.5 \) grow to very large amplitudes \( \delta n/n_0 \leq 15\% \) for \( \eta_0 > 0 \) at approximately the expected growth rate. For \( \eta_0 = 0 \), a mild instability also develops and no sign of instability was observed for \( \eta_0 < 0 \).

Figure 3 shows the density profile and the mode structure of the wave potential for \( \eta_0 > 0 \). The density profile begins changing significantly at \( \omega_{pe} t = 2200 \) where the electric field fluctuations grow beyond the noise level indicating a coherent mode structure as shown in Fig. 3. It is straightforward to evaluate \( D_{1} \) and \( \gamma/\omega \) from the measurement. We find \( D_1 = 0.013 \delta \omega_{pe}^2 \) and \( \gamma/\omega = 0.027 \delta \omega_{pe}^2 = 2D_1 \) for \( \eta_0 = 1.5 \). These numbers should be compared with the usual estimate of diffusion \( D_1 = \gamma/\omega_\perp \approx 0.3 \delta \omega_{pe}^2 \) which is much larger than the observed diffusion.

Let us now discuss the saturation of the instability for the case of \( \eta_0 = 1.5 \). As described above, the temperature diffusion is a few times as fast as that of the density. Indeed, the temperature gradient became quite small at the time of saturation at around \( \omega_{pe} t = 3000 \), which appears to have led to the saturation of the instability. No quasilinear effect was observed in the electron velocity space in contrast to the collisionless drift wave instability. In fact, it appears possible to explain the nonlinear evolution of the instability observed in the present simulation in terms of quasilinear theory for the macroscopic density and temperature including Coulomb collisions and wave scattering. While the plasma could be unstable against the finite-geyroradius instability after the flattening of the temperature, the instability appeared to have saturated. The turbulence could have quenched the weaker instability.
There are other possible mechanisms for the nonlinear saturation of this instability. One of them is the detrapping of the trapped electrons from the magnetic mirror by the turbulent electric fields. This requires an electric field potential \( \phi \) such that \( \phi / T_0 > \delta B/8 = 0.25 \) which is to be compared with the observation \( \phi / T_0 \approx 0.15 \) as shown in Fig. 3. Another is nonlinear mode coupling. However, no evidence was observed to indicate such a process. We believe that this is because the strongly unstable modes are such that \( k_y B_0 = 0.5 \) for which the propagation is highly dispersive.

While the results reported above were obtained from the 2 1/2 dimensional slab model, we have been developing a self-consistent, full three-dimensional toroidal code using a new technique for solving Maxwell's equations. In this model, the potential, for example, is expanded in terms of eigenfunction in the toroidal direction

\[
\phi(x, \theta, \phi) = \sum_n \phi_n(x, \theta) e^{i n \phi}
\]

where \( \theta \) and \( \phi \) are the poloidal and the toroidal angles. The remaining equation for \( \phi_n(x, \theta) \) is solved using the multipole expansion technique on a spatial grid.

The advantage of this model is that a full three-dimensional toroidal simulation can be carried out with the presently available computers by keeping the reasonable numbers of modes in the toroidal direction. Initial simulation results will be shown for the full three-dimensional perturbations in a toroidal system.

3. C. Z. Cheng and H. Okuda, to be published.

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Numerical Simulation of Neutral Beam Injection into a Tokamak

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Neutral beam injection is expected to play a major role in heating large-scale tokamaks together with the ohmic heating and the various RF plasma heating. In particular, the concept of TCT is to make use of the large fusion cross section of the injected beam ions while they slow down due to the interaction with the background plasmas. The heating of tokamak plasmas using the neutral beams has been successful for the existing tokamaks and appears to be promising in the future large tokamaks. The physics involved in neutral beam is highly complex; it includes the interaction due to long-range Coulomb collisions, various atomic processes including charge exchange and impurities, and beam-induced microinstabilities.

When the beam-induced instabilities or anomalous effects are neglected, the basic underlying process is classical Coulomb collisions described by the Fokker-Planck equation including the impurities and the various atomic effects. The predictions based on the Fokker-Planck equation are found to be in reasonable qualitative agreement with the experimental results. The analytical treatment is, however, so complex due to toroidal geometry and plasma inhomogeneity that a numerical approach is necessary when the detailed information is required for the design and the interpretation of the tokamak experiments.

As the beam density and the power are increased, however, the modification of the background plasma due to beam-induced anomalous effects, that is, the turbulence due to microinstabilities may not be overlooked compared with the classical Coulomb process. It is, therefore, necessary to study the anomalous beam-induced effects such as pitch angle scattering due to cyclotron instabilities and anomalous diffusion due to low frequency microinstabilities in addition to the Coulomb scattering.

For this purpose, we have developed two particle simulation codes which work on different time scales with and without including the self-consistent beam induced microturbulence.

The first of them which we call GCBEAM follows the guiding center orbits of the beam ions in a three-dimensional toroidal magnetic field. It is assumed in this model that the background plasma remains a Maxwellian defined in terms of macroscopic density and temperature and the interaction among the fast ions is neglected assuming the beam particle density is low. Self-consistent fields are also neglected in this model. The Coulomb collisions with the background plasma is built in through the Monte Carlo method as described below.

The beam-ion distribution function \( f(v_i, \theta, t) \) after the injection is governed by

\[
\frac{1}{T} \frac{\partial f}{\partial t} = \frac{1}{v_i^2} \frac{\partial}{\partial v_i} \left( \frac{v_i^2}{2} D(v_i) \frac{\partial f}{\partial v_i} \right) + \frac{1}{v_i} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta P(\theta) \frac{\partial f}{\partial \theta}
\]

(1)

where the three terms on the RHS are the energy diffusion, the slowing down and the pitch angle scattering. For the beam speed \( v_b \gg v_i \) and \( v_b \ll v_e \) where \( v_i \) and \( v_e \) are the background ion and electron thermal speed, the energy diffusion term is quite small compared with the other two terms, and we shall ignore this term with the present calculations.

Taking the first and second moments of Eq. (1), we obtain

\[
\frac{1}{T} \frac{d \langle v_i \rangle}{dt} = -\langle A(v_i) \rangle
\]

\[
\frac{1}{T} \frac{d \langle (\theta) \rangle^2}{dt} = \left( 2P(v_i) \right)
\]

We incorporate the slowing down and the pitch angle processes into our simulation code by adding two terms to the particle-pushing algorithm. For the slowing down, the change in speed, \( \Delta v \), which is given by \( \Delta v = \langle A(v_i) \rangle \Delta t \) for time step \( \Delta t \) is added to the collisionless orbit. The pitch angle scattering is provided by the same method as before.

Because of the neglect of the self-consistent field, typical step of time integration in this model is about 0.5 \( \mu \text{s} \). We can follow several thousands of beam ions up to 10 \( \mu \text{s} \) without any difficulties.

The second model which simulates beam induced microinstabilities and their effects on pitch angle scattering and the anomalous diffusion and heating is developed in order to feed the anomalous effects back to the toroidal guiding center model. The second
model is more like a conventional particle code in which the self-consistent fields and their effects on the plasmas are followed in time. Since the full three dimensional, toroidal simulation in time consuming, we adopt a cylindrical torus as before with the 2 1/2 dimensional self-consistent fields only at the present time. Using the beam ion distribution calculated from GCBEM as initial conditions, the anomalous pitch angle scattering due to cyclotron instabilities and loss-cone instabilities are calculated and compared with the classical scattering rate under various injection angles.

The Fokker-Planck collision operator modeled by the Monte Carlo method was first tested by comparing the simulation result from an independent solution of the Fokker-Planck equation. As an example, we consider a background plasma of density $n = 5 \times 10^{15}$ cm$^{-3}$, electron and hydrogen ion temperatures $T_e = T_i = 1$ keV and the total number of particles for each species $N_t = 10^{22}$. A 25 keV hydrogen beam is injected into this background plasma in a form of a very short pulse at $t = 0$. The total number of injected hydrogen atoms is taken to be $N_i = 10^{20}$.

Figure 1 shows the results from the simulation model and those from the solution of the Fokker-Planck equation. In the simulation, 1024 particles were used. Both the beam ion energy $W$ and the background ion and electron temperature $T_i$ and $T_e$ agree quite well for both calculations up to at least one slowing down time. The deviation for a later time is the breakdown of the approximation in the simulation model $v_i \ll v_e \ll v_i$. We have tested the model for several different parameters and always found good agreement with the Fokker-Planck equation.

After testing the model, we have applied it to study various problems such as velocity-space loss region and the beam induced instabilities associated with the near perpendicular injection. These results, as well as the results for different injection angles, will be presented and discussed.


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The study of the motion under the law of gravity is a key component of modern physics, and the dynamics of colliding bodies is a fundamental aspect of this field. In this context, the numerical simulation of a system of colliding bodies is a powerful tool for understanding the behavior of such systems.

The text on the page discusses the simulation of a system of colliding bodies in a gravitational field. It emphasizes the importance of accurate and efficient algorithms for simulating these systems, particularly for astrophysical applications.

The text also highlights the use of numerical methods to solve the equations of motion for these systems. These methods allow for the modeling of complex interactions and the prediction of outcomes in scenarios that are difficult or impossible to solve analytically.

The page contains several references to equations and mathematical expressions, indicating a high level of technical content. The document is likely part of a larger work on the dynamics of gravitational systems, possibly with applications in astrophysics or astrophysics applications.

Overall, the page provides a comprehensive overview of the methods and challenges involved in simulating systems of colliding bodies in a gravitational field, with a focus on the computational aspects of this field.
around the time \( t_i \) in the interval \((t_{ij}, t_{ij}+h)\):
\[
e_i(t_{ij}, t) = \frac{r_{e_i}^2}{t_{ij}} \left( \frac{t - t_{ij}}{r_{e_i}} \right)^2 + 2 \frac{r_{e_i}}{t_{ij}} \left( \frac{t - t_{ij}}{r_{e_i}} \right) + \frac{t_{ij}^2}{2} - (2t)^2,
\]
where \( t_{ij} \) and values of \( t_{ij} \) and \( t_{ij} \) are taken at the time \( t_{ij} \).

6) The exact moment at which a collision occurs is found iteratively. The basic flow of this iterative search is shown in Figure 2. The safety factors \( m_1 \) and \( m_2 \) are included in order to allow for the crudeness of the second-order expansion. A few simple tests can limit the number of collision candidates significantly.

Note that, in molecular dynamics problems, a considerable amount of the computing time and storage used in searching the "neighbours" of a given particle could be saved by sorting particles into cells. In our case, however, this method would not be useful because the number of particles involved is much smaller than in molecular dynamics. A powerful check for the detection of programming errors consists in verifying at the beginning of each step that the distance between any pair of particles has not become smaller than the sum of their radii: finding such an "interpenetration" indicates that a previous collision has been missed.

The evolution of the standard model can be summarized in the following way: after a fast flattening, contrary to what is often stated, the system reaches a quasi-equilibrium state in which collisions still occur and in which the thickness of the newly formed disc is finite — by this, we mean that the centres of the particles do not lie in the same plane. Under the combined effect of differential rotation and of inelastic collisions, the disc spreads very slowly, particles move both inwards and outwards carrying out some angular momentum. The energy, which is continually lost by inelastic collisions, is obtained at the expense of the energy of the disc which is spreading.

In the standard model, the particles were assumed to be frictionless spheres. This is of course not realistic; in real collisions there will be some friction and the grazing component of the relative velocity will be reduced after the collision. If the grazing component of the relative velocity of two colliding particles is reduced after each collision, for most cases, the evolution is similar to that of the standard model. But, for very inelastic collisions and contrary to the behaviour of the standard model, the disc is completely flattened rapidly and the system becomes a two-dimensional configuration in which collisions continue to occur.

As an example, the numerical results have been applied to real physical systems. A relation between the mean radius \( r \) of the particles of Saturn's ring (in meters) and the age \( a \) of the ring (in years) has been obtained: \( r^2 \cdot a = 3 \times 10^{11} \). If the ring is as old as the solar system, the ring's particle size cannot exceed 2.5 meters. Alfvén (1970) has suggested that planets are formed out of "jets streams". However, we see no numerical evidence for a focusing process but rather a spreading and a destruction of a stream by collisions. If a protogalaxy consists of several clouds which suffer about ten collisions per galactic revolution, it is flattened by this mechanism in about \( 10^9 \) years.
HIGH BETA EQUILIBRIA IN BEAM HEATED TOKAMAK PLASMAS

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In modern plasma confinement devices of the tokamak type, it is now possible to produce a large increase in the plasma energy density by injection of energetic neutral particle beams. Traditionally, it has been thought that equilibrium considerations would limit the plasma energy density at values such that beta (the ratio of plasma pressure nkT to magnetic pressure B^2/8\pi) would be limited to approximately 5% for plasmas of circular cross section. It has been observed, however, that the neutral injection heating technique can raise the pressure more rapidly than penetration times for currents. This suggests the possibility that current distributions could be maintained for up to the skin time, which would provide pressure balance at beta values higher than 5%.

If the initial plasma temperature (before injection heating) is sufficiently high, the plasma and magnetic fields are constrained to move together. This leads to the condition that magnetic fluxes are preserved as the heating occurs, hence the term "FCT-Flux Conserving Tokamak". Preservation of the toroidal and poloidal components \phi and \psi of magnetic flux then gives constancy of the differential ratio \eta = d\psi/d\phi, the familiar "safety factor" of the plasma \psi is a \psi the kink instability.

We have interest then to determine a quasi-static sequence of equilibrium solutions to the MHD equations \nabla \cdot \mathbf{J} = 0 and \nabla \times \mathbf{B} = 4\pi \mathbf{J}, or their reduction for axisymmetric systems independent of

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the azimuthal angle \phi in cylindrical coordinates (R, \phi, Z): \nabla \times \mathbf{J} = \nabla \times (\mathbf{F} \times \nabla \phi) = \mathbf{B}^{*} \text{ with } \nabla \phi = (\mathbf{\phi} \times \nabla \phi + \phi \nabla \phi) / \mathbf{B}^{*}.

Appropriate boundary conditions are that \phi be constant at conducting boundaries or regular at infinity. Since \eta may be written as \nabla \times \mathbf{F} \cdot \mathbf{R}^{2} \phi^{*}, this quantity is to be preserved.

It is instructive to solve first the reduced problem where the toroidal curvature of the system is neglected. This can be done analytically for the case of uniform initial current density where \eta is constant. In contrast to the situation in which \eta is not constrained, a limiting \eta is found even for cylindrical plasma, but the value \eta = \phi \text{ is sufficiently large to be ignorable.}

For a second case in which the system is still cylindrical (no toroidal curvature) but the initial current density falls off toward zero at the edge of the discharge a lower \eta \text{ limits } = 25\% 

Sample results of this are shown in Fig. 1 in which the longitudinal field and current densities are plotted for several values of \eta. The change in current behavior between \eta = 13\% and \eta = 34\% is obvious; the \eta = 28\% case is very near the transition.

For this cylindrical case it has been advantageous to transfer the FF'/4\pi term to the LHS of the equilibrium equation. A solution can then be obtained easily by a successive-over-relaxation technique for the values of \phi stored on a grid in the minor radius r.
The far more important case in which a 2 dimensional problem is solved in (R, Z) cannot readily be solved using the rearrangement approach, and it appears necessary to resort to numerical determination of \( FF' = 1/2 \frac{1}{\Psi_0} \int q_{/4}^{/2} \int \frac{d^2}{R^2 \beta_0} \). This involves a second derivative of the \( \psi \) function (since \( \beta_p = R^{-1} |\psi| \)), which creates numerical stability difficulties: the second derivative terms in \( \psi \) for the \( FF' \) are equally important to the numerical stability of iteration as is the second derivative term in LHS of the elliptic equation. We have found that the FCT condition \( q(\psi) \) preserved can be quite well satisfied in a 2D calculation by using an \( F(\psi) \) prescription obtained from the 1D procedures. Examples will be presented. The strategy here is analogous to that used by Stevens and Grad for the case of adiabatic compression in a tokamak.\(^4\)

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In Search of Stable 3D MHD-Equilibria

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Abstract
A possible way of finding 3D MHD-equilibria is to time-integrate a modified system of MHD equations where inertia has been replaced by a fictitious friction (MHD-model)\(^{\text{1,2}}\). Starting from a variational formulation the discretisation of the equations is described by several methods. The admissible time-step of the explicit integration is improved by a factor of about \(10^3\) over the usual stability condition. The method is used to look for the final state of unstable screw-pinches and tokamak type equilibria.

Variational formulation
A magnetohydrostatic equilibrium is described by the scalar field \(\rho(x)\) and the vector field \(B(x)\) which fulfill the requirement
\[
\Delta W = 0
\]

\[\Delta W = \int \left( \frac{1}{2} B \cdot B + \frac{1}{\gamma - 1} \rho \right) \, dt \quad \text{(1)}
\]

The variations of the fields are constrained to displacements by an infinitesimal vector field \(\xi(x)\) such that
\[
\delta B = -\text{div} \xi \quad \text{and} \quad \delta B = \text{curl} \xi \times B
\]

\(\gamma\) is the ratio of specific heats; homogeneous specific entropy has been assumed; \(\text{div} B = 0\) is required and compatible.) Boundary conditions are vanishing normal components of \(B\) and \(\xi\) on a surface which is assumed to form a toroid. If \(W\) is not only stationary but a (local) minimum under the constraints (2) the equilibrium is called stable.

Inserting (2) into (1) we define the force density \(F(x)\) by
\[
\Delta W = -\int F(x) \cdot \xi(x) \, dt \quad \text{(3)}
\]

resulting in the familiar expression for \(F\)
\[
F = -\gamma \rho \hat{B} + \text{curl} B \times B
\]

We define then a time-like variable \(t\) by writing
\[
\xi = \gamma \, dt \quad \Delta W = \hat{W} \, dt \quad \text{etc.}\]

The relation
\[
\gamma(x,t) = \gamma(x,0) + \frac{1}{2} \Delta W \quad \text{(5)}
\]

(together with (2) and (5)) form for any positive definite \(\gamma\) a first order vector partial differential equation describing the evolution of \(B\) and \(\rho\). It differs from the familiar ideal MHD equation by the replacement of inertia by an artificial friction \(\gamma\) being the friction coefficient. We therefore propose to call it the MHD model.

\(^{\text{1}}\) R. Schlüter, Sitzungsberichte der Bayer. Akademie d. Wissenschaften 2 R. Chodura, A. Schlüter, 3rd Top.Conf.on Pulsed High Beta Plasma, Culham, Sept. 1975

From (3) follows
\[
\hat{W} = -\gamma \rho \Delta W < 0
\]

Since \(W\) by its definition cannot decrease indefinitely the equilibrium condition \(\hat{W} = 0\) must be approached. The resulting final \(W\) may be a minimum except if both the boundary and the initial conditions allow symmetry operations. Barring this case, any MHD-solution develops into a MHD-stable equilibrium. This equilibrium might show unwanted properties, and in particular the plasma described in this way might not be sufficiently removed from the material wall described by the boundary.

Spatial discretisation
The preceding way of introducing MHD lends itself to a natural discretisation of the spatial dependence: The integral in the definition (1) of \(W\) is replaced by a sum over the grid points (including weight factors, if the partial volumes represented by different grid points vary). The constraints (2) are likewise replaced by their discrete analogues. On the boundary, the absence of matter and magnetic flux through the surface is needed. It turns out to be also sufficient, if the grid points, on which \(\gamma\) is evaluated contain the surface while those of \(B\) and \(\rho\) are displaced by half a mesh width in all three directions. Eq. (3) or rather its discrete analogue renders the appropriate discretisation for \(F\) (inclusive of boundary conditions) such that (7) again holds and convergence towards a stable solution is again secured.

Discretisation in time
Natural units of the physical quantities are employed such that the minor radius of the toroid = 1 and that the Alfven speed = 1. In these units, the time scale in which the (minor) radial compression (or expansion) takes place if initially gross non-equilibrium exists has a time scale of order \(1\) and faster. Interesting MHD instabilities of the then obtained near-equilibrium may extend down to timescales of the order \(10^3\) to \(10^{-2}\), say. The well-known critical time step \(\Delta t\) for the standard explicit integration scheme is of the order \(\Delta x^2\), where \(\Delta x (\ll 1 \text{ in natural units})\) is the mesh width. Instead of using basically different schemes, we have accelerated the explicit scheme by a) using sequences of varying time steps, b) replacing \(\gamma\) by a positive definite non-local anisotropic linear operator. The gain achieved corresponds for the longest part of a typical integration to almost a factor \(10^3\) (sometimes also more).

To a): If in a cycle of \(n\) time steps the \(n\)-th is given the length \(\Delta t_n = 2 \alpha \cos(2\pi -1) n^2/\alpha^2\) where \(\alpha > 1\), then this cycle ("Superstep") leads to a stable integration, provided integration with equal steps of length \(\Delta t\) were stable. This can be proven, provided linearization of the differential equation over the different time steps is admissible and leads to a self adjoint system. For the limiting case \(n = 1\), the length of the superstep is exactly \(\Delta t\) and the average of the single steps is correspondingly \(\alpha = 2\). Positive values of a slightly in excess of 1 are chosen rendering the procedure more stable and relaxing somewhat the condition of strict selfadjointness. The gain is then of course somewhat reduced. In a standard case with \(n = 101\) a typi-
cal gain is \( g = 60 \). Supersteps with \( n = 307 \) and a gain of \( g = 180 \) have under favourable conditions been used successfully.

To b): The eventual convergence towards a stable solution is maintained if in eq. (6) \( \lambda \) is replaced by any positive definite operator (operating on the space variables). Of this possibility twofold use is made: 1) \( \lambda \) is replaced by a (local) anisotropic tensor, such that the friction is less in that direction in which the grid mesh has the longest mesh size, or such that the friction is less along the magnetic lines of force (by a factor of the order \( B^2 \)). The potential gain of this procedure will be particularly large when \( B \) is small, it depends much on the prevailing excited modes; the presently obtained gain is estimated to be a factor of 2. 2) \( \lambda \) is replaced by a "smoothing" operator of the convolution type. This diminishes the damping rate of the short wave length and therefore increases the value of \( g \). The achieved gain is a factor of 8 to 10. Care must be taken at the boundary; it has turned out that the parallel component of velocity on the boundary should not be smoothed in the normal direction.

Preliminary results

We have used the MIF model to study the behaviour of screw-pinch and tokamak type equilibria. The calculation volume has a quadratic cross section of side width \( L \) and length \( L \) either straight or bent into a torus of aspect ratio \( A \). The inverse rotational transform at the center is \( q_0 \).

Fig. 1 shows a schematic drawing of the time evolution of a run. The spatial mean square value of the force density \( f^2 \) is plotted as a function of time. We start with a near equilibrium distribution of density and poloidal field. After a period of relaxation (towards \( B_{pol} = 1 \) plasma) the axial magnetic field is switched on and at the same time a sequence of 10 stochastic disturbances of density sets in separated by one superstep of relaxation. This disturbance breaks the symmetry of the initial conditions and makes the plasma free to perform 3D unstable motions. If the plasma is stable the disturbance dies away and equilibrium is reached again.

Figs. 2a-d show the behaviour of a straight plasma column for decreasing values of \( q_0 \). The pictures to the left are computer prints of the isobares in three perpendicular planes, the curve to the right indicates the deviation from equilibrium. In particular, a rise of \( \langle f^2 \rangle \) after a minimum indicates an instability. The times of the pictures are marked by strokes at the upper line of the \( \langle f^2 \rangle \)-frame.

Fig. 2a with \( q_0 = 4 \) shows a nearly stable situation (the weak rise of \( \langle f^2 \rangle \) at very late times is possibly due to an unphysical grid oscillation). Figs. 2b-d show pronounced unstable modes with axial mode number \( n = 1 \) and poloidal mode number \( m = 3, 2 \) and 1 respectively. The \( m = 1 \) mode finally touches the wall.

Fig. 2e shows the time-history of a toroidal case with \( q_0 = 1 \) (only 1/4 of the total torus were covered by the computation grid). Here the fast radial equilibration is followed by a fairly long period of toroidal drift. Then again the disturbance is switched on and triggers a \( n = 1, m = 2 \) instability as in the linear case.
A TWO-DIMENSIONAL MHD STABILITY CODE WITH A FINITE HYBRID EXPANSION IN BOTH DIRECTIONS

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ABSTRACT

A new MHD stability code which computes the spectrum and the corresponding eigenvectors of a general toroidal axisymmetric equilibrium is described. It is based on a "finite hybrid elements" expansion. Test runs show very good convergence properties even for the slow growing localized modes.

The ideal MHD spectrum of an axisymmetric toroidal equilibrium can be obtained by variation of the Lagrangian \( L = 6w - \frac{1}{2} \left[ \frac{d}{dt} (\mathbf{B} \times \mathbf{E}) \right] ^2 \), where \( 6w \) is the usual expression for the potential energy and \( \mathbf{q} \) is the displacement. Using the non-orthogonal coordinate system \((\mathbf{q}, \chi, \phi)\) introduced by the Princeton group [1], and Fourier expanding in the \( \mathbf{q} \) direction \( \mathbf{q} = \text{Re} \left( \mathbf{Q}(\chi, \phi)e^{-i\phi} \right) \), the plasma contribution to \( 6w \) [2] can be rewritten as:

\[
6w = \int \left[ \frac{dx}{x} \frac{d\chi}{\chi} + \frac{dx}{x} \frac{d\phi}{\phi} \right] |F(\chi, \phi)|^2 + \frac{3}{4} \left[ \frac{\chi}{\chi_0} \frac{x}{x_0} \left( \frac{\chi}{\chi_0} \frac{x}{x_0} \right)^2 \right] F(\chi, \phi) + \frac{2}{q} F(\chi, \phi) + \frac{1}{q} \int F(\chi, \phi) \right)^2 - \frac{3}{4} \left( \frac{\chi}{\chi_0} \frac{x}{x_0} \right)
\]

where

\[
\begin{align*}
F &= i \mathbf{q} - \frac{2}{x} \chi^2 - \frac{2}{r} \chi^2 \frac{d}{dx} \left( x \frac{d}{dx} \right) \chi^2 + \frac{d}{dx} \chi^2 \left( \frac{d}{dx} \chi^2 \right) \chi^2, \\
\chi^2 &= \frac{2 \chi_0}{\chi_0} - \frac{1}{2} \left( \frac{\chi}{\chi_0} \frac{x}{x_0} \right)^2, \quad \chi_0 = \frac{2 \chi}{r} \frac{d}{dx} \left( x \frac{d}{dx} \right) \chi^2, \quad \chi_0 \quad \text{and} \quad \chi_0 \quad \text{are}
\end{align*}
\]

the poloidal and toroidal components of the magnetic field respectively, \( j \) the toroidal current density, \( q(\chi) \) the safety factor and \( r(\chi, \phi) \) the distance to the main axis of the torus.

A 2-D code which uses a finite element expansion of the Lagrangian has been already described [3]. It uses Mercier's form for \( 6w \) in an orthogonal coordinate system. By an adequate choice of the basis functions, such that \( \mathbf{q} \cdot \mathbf{B} = 0 \) can be satisfied identically over a mesh cell, external unstable kink modes are reproduced with a good accuracy for any value of \( \beta \). This has been verified using Gajewski's equilibrium [4] with a small ellipticity. The convergence though deteriorates as the ellipticity increases, probably because of the pathological behaviour of the equilibrium quantities near the magnetic axis introduced by the orthogonal coordinate system. Weakly growing modes localized around \( \psi \approx 1 \), cannot be found as accurately with a reasonable number of elements even at moderate values of \( \beta \), and for an ellipticity above 1.2 no unstable modes are found. The difficulty with the slow growing modes is that their growth rates are higher order in the Tokamak ordering. With our choice of elements the first term \( |F(\chi, \phi)|^2 \) in \( 6w \) is not represented with a sufficient accuracy, the error on each cell being of the same order as the kink.

To remedy this situation we have constructed a new code which uses instead of [2] the new expression (1) for \( 6w \), and in which \( \chi \) is approximated by a so-called "finite hybrid elements" expansion [5]. Since the localized modes are the most difficult to reproduce we consider the fixed boundary problem with \( 6w = 6w_\beta \), imposing the condition \( \chi = 0 \) at the plasma surface. On the magnetic axis we only impose \( X = 0 \), as the regularity condition between \( X \) and \( V \) comes out naturally as consequence of the \( \chi^2 \) factor in \( 6w \). With the choice of variables made, all quantities are now regular at the origin. To discretize the Lagrangian we cover the domain \( 0 \leq \psi \leq \chi_0, 0 \leq \chi \leq 2\pi \) with a rectangular mesh. \( X \) is defined at the mesh points \( i,j,k \) and \( Y,V \) at the displaced points \( i+1,j \), \( k \), as shown in Fig. 1. In the integral (1), on each cell, \( X/\chi, X/V, Y/X, V/\chi \) are taken as constants defined in terms of the values of \( X,Y \) and \( z \) on the edges of the cell by:

\[
\begin{align*}
X &= \frac{X_{i+1,j} + X_{i,j+1} + X_{i+1,j+1} + X_{i+1,j+1}}{4}, \\
\frac{\delta X}{\delta \psi} &= \frac{X_{i+1,j} + X_{i,j+1} + X_{i+1,j+1} - X_{i,j} - X_{i,j}}{2\delta \psi}, \\
\frac{\delta X}{\delta \chi} &= \frac{X_{i+1,j} + X_{i,j+1} + X_{i+1,j+1} + X_{i+1,j+1}}{2\delta \chi}
\end{align*}
\]

Figure 1

\[
\begin{align*}
X &= \frac{X_{i+1,j} + X_{i,j+1} + X_{i+1,j+1} + X_{i+1,j+1}}{4}, \\
\frac{\delta X}{\delta \psi} &= \frac{X_{i+1,j} + X_{i,j+1} + X_{i+1,j+1} - X_{i,j} - X_{i,j}}{2\delta \psi}, \\
\frac{\delta X}{\delta \chi} &= \frac{X_{i+1,j} + X_{i,j+1} + X_{i+1,j+1} + X_{i+1,j+1}}{2\delta \chi}
\end{align*}
\]
The equilibrium coefficients are also replaced by piecewise constant functions having the exact value at the center of the cells. Performing the remaining trivial integrations the problem reduces to a matrix eigenvalue problem

\[ AX - \omega^2 BX = 0 \]

where A and B are symmetric. The block structure of these matrices allow us to use the fast matrix solving programs HYNIA-BLOCK [5].

With this scheme all the terms in 6 P are constant over a cell and thus can cancel exactly. This is expected to remove the problems encountered with the regular finite elements method.

**NUMERICAL TESTS**

To test this new code we choose a simple analytical toroidal equilibrium characterized by \( T^\prime = 0 \), \( j = j_0 r/R \) (\( R = \) radius of the magnetic axis) and \( E \):

\[ \psi = \frac{E^2}{E^2 + 1} \frac{J_0}{2R} \left( \frac{r^2}{R^2} + \frac{(r^2/R^2)}{2} \right). \]

In the limit of a large aspect ratio, it reduces to the elliptic Gajewski's equilibrium, \( E \) becoming the ratio of the vertical to the horizontal axis of the ellipses. All the tests shown here are done with a mesh equidistant in \( \psi \) and \( \chi \); \( N_\psi \) and \( N_\chi \) denote respectively the number of intervals in the \( \psi \) and \( \chi \) direction. We have used the up-down symmetry to reduce the range in \( \chi \) to \( 0 < \chi < \pi \).

We first look at the special case of a circular cross-section (\( E=1 \)), in the limit of large aspect ratios (\( R = \infty \)). Fig. 2 shows a comparison of the growthrates of the first three \( m=1 \) unstable modes obtained respectively with a 1-D code (-), with the finite elements 2-D code (o) and with the new hybrid code (a). The normalizing growthrate \( \omega_A \) is the Alfvén frequency

\[ \omega_A = \frac{a}{\rho a R}, \]

where \( a \) is the plasma radius.

The current density chosen corresponds to a \( \delta_{axis} \) of .08. The results with the regular finite elements become worse as \( \delta \) decreases. We see that the hybrid method gives very good results. If one ignores the shift in \( n_q \) of the position of the maximum growthrate, which is so small as to be physically meaningless, the agreement is even better.

**REFERENCES**


AXISYMMETRIC STABILITY OF NUMERICALLY COMPUTED TOKAMAK EQUILIBRIA

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Stability against axisymmetric modes is a major design consideration for tokamaks using applied external currents for shaping the plasma cross-section or for producing an axisymmetric divertor. Codes have previously been developed for testing a given equilibrium configuration against rigid vertical displacement /1/,/2/. Recently Rehban and Salat /3/ have found higher-n axisymmetric modes to be unstable in a surface-current model for plasma cross-sections with a triangular or rectangular deformation. The latter instability has also been found by Rosen /4/ for a constant plasma current distribution, but not in a corresponding experiment in the G.A. octupole /5/. This discrepancy with experiments could be due to a strong influence of the actual plasma current distribution, and underlines the importance of a code testing arbitrary, numerically computed equilibria against general axisymmetric modes.

Such a code has been developed using as input the results of the Garching equilibrium codes. Essential features of the stability code are:

1) $\delta W$-analysis, giving frequencies or growth rates for all modes included in the calculation
2) slip-motion ansatz of Rehban and Salat /3/ for the permitted displacements
3) a non-orthogonal coordinate system using flux surfaces ($x^1$ = const.) and a second coordinate chosen to make $B^1/B^2 = q(x^1)$
4) representation of the expansion functions by a truncated Fourier series in $x^2$ and finite elements in $x^1$
5) determination of the vacuum contribution to the energy integral by Martensen's integral equation method /6/.

Points 3 and 4 have been first proposed and tested by the authors of the Princeton general stability code /7/, who have also pointed out the advantages of investigating the complete spectrum rather than only the unstable modes.

The slip motion ansatz describes modes doing no work against the applied toroidal field, and thus yields stability boundaries independent of $q$. For the surface current model a noticeable difference between these and the true axisymmetric stability boundaries has been found only for aspect ratios < 3 /8/. Using the slip motion ansatz, we substitute the displacement vector $\tilde{x}$ by a "quasi-stream-function" $\tilde{q} = \tilde{z} + \tilde{v} x$.

The choice of coordinates (3) makes the plasma-vacuum interface a coordinate surface, simplifies some algebraic expressions and improves the resolution in the "corners" of the configuration, where the $x^2 = \text{const.}$ lines tend to concentrate. Moreover, it improves the representation of the physical spectrum, which in our case consists of slow (acoustic) modes propagating along $x^1 = \text{const.}$, and modes propagating with $\sqrt{\nabla P - \nabla P_{\text{pol}}} / \sqrt{\nabla P_{\text{pol}}}$ parallel and perpendicular to the flux surfaces.

In analogy to /7/, $X$ is represented by a linear superposition of expansion functions, chosen as $2 M + 1$ Fourier modes $e^{i \mu x^1}$ and $N$ tent functions in $x^1$. Inserting this expansion for $X$ into the expressions for $\delta W$ and the kinetic energy $K$ and carrying out the integrations over the plasma region converts the energy principle into a matrix eigenvalue problem.

As the slip motion ansatz and the finite number of terms in the expansion limit the class of the allowed displacements, we can interpret results strictly only as yielding a necessary condition for stability. To be, however, at least consistent with this claim, we have to compute the equilibrium quantities appearing in $\delta W$ and the integrals with high accuracy, even for the simple expansion functions chosen by us, and for small values of $N$ and $M$.

We therefore compute the equilibrium quantities also in
intermediate flux surfaces between $x_n^1$ and $x_{n+1}^1$ and carry out the integrations over $x^1$ using interpolation polynomials for the equilibrium quantities and Gaussian integration. For the same reason the Fourier analysis of the equilibrium quantities is carried out up to order $m = 2N$, the highest order contributing to the integrals over $x^2$.

Examples for which these features turn out to be crucial will be given at the conference.

The vacuum contribution to $\delta W$ is computed by using the fact that for any given displacement at the boundary, the energy of the vacuum perturbation field is minimized by the corresponding solution of Maxwell's equations. This solution is obtained by using Martensen's integral equation method, which takes into account the multivaluedness of the scalar potential in the axisymmetric case, and removes the singularities from the integral equations /6/.

Figure 1 gives results for elliptic plasma configurations with large aspect ratio ($b/R = 10.3$) and a plasma current distribution $I_p(r)$, for which rigid displacement is expected to be most unstable mode. The frequency $\omega$ is normalized by $\omega = \frac{\omega_p}{I_p^{1/2}/p R^2}$ with $I_p^{1/2}/p$ the total plasma current. The part of the spectrum shown includes the slow waves $\omega_p < 7.5$ and that part of the fast waves that eventually becomes unstable. The dotted line gives the results for rigid vertical displacement (computed using the code described in /2/) showing excellent agreement.

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"This work was performed under the terms of the agreement on association between the Max-Planck-Institut für Plasmaphysik and EURATOM".
MHD INSTABILITIES OF 2D DIFFUSE HIGH-BETA EQUILIBRIA AS AN INITIAL-
BOUNDARY-VALUE PROBLEM

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Abstract: A time-dependent explicit finite difference code is used for computing very small
growth rates of gross MHD modes in helically symmetric equilibria. The problem of numerical
dispersion and the convergence for fine grids is discussed. For equilibria with long helical
period length, the eigenvalues computed by this code are in agreement with those of a
\( \Delta W \) analysis and the experiment, and are by a factor of two smaller than surface current
theory predicts.

1. Introduction. The gross modes of MHD instabilities can be calculated by solving the
ideal MHD equations as an initial boundary value problem. This problem has been treated
for axisymmetric equilibria (Tokamak and Belt Pinch) using a linearized model \( \lambda_1, \lambda_2 \);
extensions towards the non-linear problem are tackled by several authors \( \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \).
Another possible extension of the linearized model for the general class of helically symmet-
ric equilibria \( \lambda_8 \) is of great interest for high-beta stellarator experiments \( \lambda_9 \) (HTS at Gar-
ching and SCYLLAC at Los Alamos). These experiments have a large aspect ratio and many
periods around the torus, so that the stability property of this configuration is not much
affected by toroidal effects. In contrast to the axisymmetric equilibria, these configurations
have the property that the longitudinal current vanishes on every flux tube. Consequently,
the instabilities are no longer current driven but pressure driven modes, for which the growth
rate (in units of the Alfvén transit time) is an order of magnitude smaller. Hence the in-
fluence of numerical dispersion gets more important, and considerable effort has to be spent
to get reliable results. The paper will describe the difficulties encountered and give a
brief survey on the results obtained by this code.

2. The Model. The global modes under consideration are described by the ideal MHD equa-
tions for the velocity \( \mathbf{v} \), the pressure \( p \), and the magnetic field \( \mathbf{B} \). Given a static helically
symmetric high-\( \beta \) equilibrium with arbitrary pressure profile, described by \( \mathbf{v} = 0, \mathbf{\rho}, \) and \( \mathbf{B} \),
we linearize the equations about this 2D equilibrium. The problem is formulated in a non-
orthogonal coordinate system \( \{x^i\} \), which consists of a cartesian \( \{x^1, x^2\} \)-plane rotating
around the helically deformed \( z = x^3 \) coordinate axis with the period length \( \lambda_\Omega = 2\pi/\Omega \) (a
field \( \mathbf{B} \) = 1 field (alpha. plasma radius)). We Fourier-analyze the linearized
equations along the ignorable coordinate \( x \) with a wave number \( K \).

\[ U(x^1, x^2, x, t) = \Re \{ u(x^1, x^2, \lambda K x) \} \]

where \( u(x^1, x^2, t) \) are complex quantities in order to describe the phase of the pertur-
bation. Because the \( (x^1, x^2) \)-plane rotates around \( z \), the wave number \( K \) is associated with
a set of \( (k, m) \)-modes in a frame at rest like \( K = k - m \Omega \), where \( m \) is the azimuthal mode
number.

The system of equations for the perturbed state \( \mathbf{v}, p, \mathbf{B} \) is written in co- and contra-
variant vector notation:

\[ \mathbf{\hat{g}} \cdot \mathbf{v} + \mathbf{g} \cdot \mathbf{v} + m \mathbf{\hat{g}} \cdot \mathbf{B} = \mathbf{0}, \]
\[ \frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \mathbf{\hat{g}} + \mathbf{g} \mathbf{v} + x \mathbf{\hat{g}} \frac{\partial}{\partial x} \mathbf{v} + \mathbf{g} \mathbf{v} \mathbf{\hat{g}} = \mathbf{0}, \]
\[ \frac{\partial \mathbf{B}}{\partial t} = \mathbf{v} \mathbf{\hat{g}} + \mathbf{g} \mathbf{B} = \mathbf{0}, \]
\[ \frac{\partial \mathbf{g}}{\partial \mathbf{g}} \mathbf{g} + \mathbf{g} \mathbf{g} \mathbf{g} = \mathbf{0}, \]
\[ \frac{\partial \mathbf{v}}{\partial x} \mathbf{g} + \mathbf{g} \mathbf{v} \mathbf{g} = \mathbf{0}, \]
\[ \frac{\partial \mathbf{B}}{\partial x} \mathbf{g} + \mathbf{g} \mathbf{B} \mathbf{g} = \mathbf{0}, \]
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In general, any arbitrary perturbation may be used as initial condition; we choose either
a velocity distribution roughly resembling a particular mode, or take the whole eigen-
function for that mode from a similar run.

The equations for the perturbed quantities are solved using an explicit difference scheme
on a grid staggered in space and time. The solution of this initial boundary value problem
yields the time development of the instability from which the growth rate is cal-
culated.

3. Convergence of the Calculation. When running the code for standard parameters
\( (a^2 = 1, h, a = 0.1, \beta = 0.5, \) no longitudinal net current) on a rather coarse grid (24x24
mesh points), all the modes \( m = 1, 2, 3 \) have been found to be stable. Therefore we tried
equilibria with a longitudinal current and found in accordance with theoretical results unstable modes. Then we tried to approximate the marginal stability behaviour of the theta-pinch; we got again stability, but when varying the number of gridpoints, we found a dependence of the growthrate on the gridsize, which is shown in Fig. 1. The extrapolation for $\Delta x \to 0$ gives marginal behaviour $\gamma' = 0$.

Having done all that, we tackling again the initially posed problem just varying the gridsize, i.e. going up to as much as $64 \times 64$ grid points. The result of this variation is plotted in Fig. 2. Again the extrapolation for $\Delta x \to 0$ seems to give reliable results; the convergence of the eigenvalue $\gamma^2$ goes roughly like $(\Delta x)^2$. But due to the dispersive effect of the finite gridsize, one has to use rather fine meshes in order to obtain the convergence curve.

4. Results. Because the rather high effort in computing a couple of e-folding times of the time evolution of the instability prevents a detailed parameter variation, we only could do a limited series of cases which are of particular interest for the HBS experiment at Garching. The first variation was on $\beta$; this result was also important for the comparison with surface current theory /11/ and a $\nabla W$ analysis for diffuse pressure profile /10/. It turned out that the result of the initial boundary value problem agrees very well with the latter, and, gives growth rates which are by a factor of 2 smaller than those of the surface current model. This is shown in Fig. 3.

Another variation was the increase of the helical displacement $\delta_e$ of the magnetic axis; again a comparison with the other models is made, and again the code yields reduced growth rates. This is shown in Fig. 4. The experimental results on the Garching ISAR T1-B experiment /9/ in the collision dominated regime also agree rather good with the numerical results.

5. Conclusions. The method described here is a useful tool for studying MHD instabilities of 2D equilibria, even for small eigenvalues. Besides a couple of extensions in the parameter space $(h,k)$, an improvement of the numerical scheme is desirable; i.e. the dispersive effect of the difference approximation should be reduced and the solution method should be speeded up. Another problem of physical interest is the stabilization by finite Larmor radius effects; this extended model requires an implicit solution method.

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Analytical and Numerical Study of MHD Instabilities in Rotating Plasmas

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Since plasma rotation is observed in confinement experiments, we investigate the normal modes in rotating straight Theta pinches of infinite length. It is our objective to obtain a qualitative picture of the structure of the spectrum in the complex plane, and to determine the growth rates of gross instabilities quantitatively.

In cylindrical coordinates \((r, \theta, z)\), the equilibrium magnetic field and plasma velocity are given by \(B = B(r)e_z\), \(u = r\Omega(r)e_\theta\), and the pressure balance is

\[
\frac{d}{dr}(P + \frac{1}{2} B^2) = \rho \nabla^2 r.
\]

For perturbations \(\propto \exp[i(-\omega t + m\theta + kz)]\), the linearized equations of motion can be reduced to a system of two first order ordinary differential equations for \(u \propto u^{(1)} + kr^{(1)}\) and \(v \propto v^{(1)} + br^{(1)}\):

\[
F \frac{d}{dr} \begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix},
\]

where both \(F\) and the matrix \(M\) are functions of \(\omega, k, m,\) and \(r\).

The spectrum of eigenvalues \(\omega\) is determined by the boundary conditions \(u = 0\) at \(r = 0\) (regularity) and \(r = R\) (perfectly conducting rigid wall). Since the problem (2) is highly nonlinear in \(\omega\), the structure of the spectrum is not known a priori. In order to develop a code which need not search the entire complex \(\omega\) plane and which assigns mode numbers without plotting eigenfunctions, we derive properties of the spectrum analytically. It can be seen by inspection that the continuas (given by those values of \(\omega\) for which \(F\) has zeros for real \(r\)) are confined to the real axis, that the spectrum is symmetric to the real axis, that it is unchanged if \(k\) is replaced by \(-k\), and that it is reflected at the origin if \(m\) is replaced by \(-m\). As a consequence, the unstable spectrum is discrete, and we can restrict attention to the upper half plane and to positive mode numbers \(m\) and \(k\).

The problem (2) depends on the ratio of rotation energy and magnetic energy, measured by the parameter \(\varepsilon^2 = \frac{e_0^2 e_0^2 \rho}{\beta_0}\) (subscripts zero denote characteristic values). Since \(\varepsilon^2 \approx 10^{-4}\) in experiments, an expansion in \(\varepsilon^2\) should yield good results. For \(\varepsilon^2 = 0\) we have a static Theta pinch, which is stable, and whose marginal modes are characterized by \(\omega = 0\) and \(k = 0\). This suggests to non-dimensionalize according to \(\omega = m\Omega(\bar{\omega})\), \(k = \bar{k}/\Omega(\bar{\omega})\), \(\Omega = \Omega(\bar{\omega})\), thus considering a singular perturbation of the nearly marginal modes. For \(\varepsilon = 1\) (rigid rotation) and \(\varepsilon^2 = 0\), the system (2) then is equivalent to

\[
(A \alpha') = [\mu_0 A/r + \rho'(2\alpha - 1)]\alpha ,
\]

where \(A = -(\omega - 1)^2 + kr^2\), primes denote derivatives with respect to \(r\), and bars have been omitted. The problem is now greatly simplified in that it is only quadratic in \(\omega\). When multiplied with the complex conjugate of \(u\), and integrated with respect to \(r\), Eq. (3) implies that complex eigenvalues can be present only in the annulus bounded by the two concentric circles with radii \(\frac{1}{2} \sqrt{1 - 4k^2} \) and center at \(\omega = 1/2\), where \(\sqrt{2} = B^2/\rho\). Obviously, each complex eigenvalue is on the circle with radius \(1/2\) and center
at $\omega = 1/2$ if $k = 0$. If $k$ is increased, it moves into this circle, tracing out a curve which reaches the real axis for some critical value $k_c (k^2 \leq 1/4 \nu_{\text{min}}^2)$. For $k \geq k_c$ it is real.

For $k = 0$, Eq. (3) takes the form

$$-(\rho u')' + \frac{m^2 \rho}{r} u = -\lambda \rho u, \quad \lambda = \frac{1-2\omega}{(m-1)^2}. \quad (4)$$

This is a Sturmian eigenvalue problem with real eigenvalues $\lambda$. If these are ordered according to their magnitude, $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$, the eigenfunction associated with $\lambda_n$ has exactly $n$ zeros in the interval $0 < r < R$. In other words, $n$ is the radial mode number.

It can be shown that $\lambda_0 \geq m^2$. As a consequence, the corresponding eigenvalues $\omega_n$ are all on the above circle (i.e. there are no real eigenvalues for $k = 0$), $\text{Re} \omega_{n+1} > \text{Re} \omega_n$, and $\omega_n \to 1$ as $n \to \infty$.

Using comparison theorem we find that each $\lambda_n$ increases if $m^2$ is increased or $R$ is decreased. $\omega_n$ then moves to the right, and the growth rate $\text{Im} \omega_n$ increases or decreases depending on whether $\lambda_n < 2$ or $\lambda_n > 2$. Since $\lambda_n > m^2$, the growth rate can increase with decreasing $R$ only for small $n$ and $m = 1$ (gross kinks).

We can now state that each complex eigenvalue (with $\text{Re} \omega > 0$) is uniquely characterized by the three mode numbers $m$, $n$, and $k$: $m$ determines the circle with radius $m\omega/2$ and center at $m\omega/2$ (here we use the original dimensional variables), $n$ determines a curve connecting some point on this circle with the real axis, and $k$ determines the location of the eigenvalue on this curve. Note that the radial mode number has been assigned by continuity, even though this need not count the number of nodes for $k \neq 0$ (the eigenfunctions are complex).

Our numerical procedure follows the same pattern: We first compute $\lambda_n$ from Eq. (4) (this is straightforward because of the Sturmian character of this problem), thus obtaining $\omega_n$ for $k = 0$. Then we increase $k$ in sufficiently small steps, each time solving the eigenvalue problem (3) with the previously computed eigenvalue as a first guess, using the shooting method and linear interpolation. This works well for small $n$ (which are the only modes of practical interest); for large $n$ it fails because the eigenvalues then accumulate at $\omega = m\omega$.

The same method works for finite values of $\varepsilon^2$ and for non-rigid rotations. Even though the $\lambda_n$ do not yield the $\omega_n$ for $k \neq 0$, they still yield suitable guesses. It turns out that increasing $\varepsilon^2$ to realistic values hardly changes the curves traced by the eigenvalues for small $n$ (their location on the curves changes drastically because $k$ scales with $\varepsilon$). Rotation shear, however, appears to shift these curves strongly.

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THREE-DIMENSIONAL, NONLINEAR MAGNETOHYDRODYNAMIC COMPUTATIONS OF THE POSTIMPLUSION DYNAMICS OF THE LOS ALAMOS SCYLLAC EXPERIMENT

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The Scyllac high-beta stellarator experiment being conducted at Los Alamos is designed to produce high beta plasmas in toroidal equilibrium by adding $\ell = 0$ (bumpy) and $\ell = 1$ (helical) perturbations to the basic theta pinch fields. The perturbations are produced by shaping the inner surface of the compression coil which drives the theta pinch implosion. The imploded plasma accommodates itself to the resulting perturbed field by assuming a bumpy, helical shape.

The Scyllac experiment is being studied by means of the numerical solution of nonlinear, time-dependent equations with appropriate boundary conditions. These calculations address questions which are not answered definitively by either analytic theory or experiment. For example, the existence of three-dimensional equilibria can be investigated by solving an initial value problem in which the state of the plasma at every time is physical. Alternative time independent formulations not only assume the existence of equilibria, which is still an unresolved question, but also do not recognize the intrinsic, transient behavior of the high beta experiment.

Numerical calculations of the post-implosion phase of the Scyllac experiment have been performed. In these calculations, the motion of a dense plasma column embedded in a field is computed by a composite algorithm which resolves the moving interface between the plasma and the vacuum. The equations for ideal magnetohydrodynamic flow describe the plasma dynamics, and the appropriate potential functions describe the magnetic field between the inner surface of the shaped coil and the plasma vacuum interface. The interior of each region is resolved by the arbitrarily shaped, six-sided cells of the arbitrary Lagrangian-Eulerian computation mesh.

Interior to the plasma-vacuum interface, the nonlinear equations for three-dimensional MHD flow of the plasma are solved using an algorithm described by Brackbill. In this algorithm, generalized spatial difference equations are derived for an arbitrary mesh by coordinate transformation. Forward time differentiating results in an implicit system of difference equations which are stable for all signal speeds. Truncation errors in the time differentiating add a diffusion to the equations which depends only upon the signal speed and the time step. A limitation on the size of the time step is determined by the number of iterations required to solve the implicit equations guarantees that all important signals are resolved and, thus, undamped.

The moving interface between the plasma and the vacuum is resolved by a surface of the computation mesh which is constrained, at every point, to move with the local plasma velocity.

Exterior to the interface, the vacuum magnetic field is computed from a (multivalued) scalar potential. The multivalued potential of the electric field is labeled by the indices $i$ and $j$, which label the vertices of the mesh in the poloidal and toroidal directions respectively. The magnetic induction, $B$, is given by,

$$B = \nabla \psi \left( w_i + w_j + \psi \right),$$

where $w_i$ and $w_j$ are constants, and $\psi$ is the periodic (single-valued) part of the potential. The periodic potential, $\psi$, is determined by minimizing the integral, $I$,

$$I = \int \frac{1}{2} \left| \nabla (w_i + w_j - \psi) \right|^2 \, dt,$$

subject to Neumann conditions at the boundaries of the vacuum region. The constants $w_i$ and $w_j$ are determined by specifying the toroidal and poloidal magnetic fluxes and the boundaries of the vacuum region. These constants are determined by finding the flux associated with two independent estimates of their value. From these trial solutions, the linear combination which corresponds to the specified flux can be obtained. The integration of $I$ over non-conforming elements, as used by Brackbill, results in a system of linear difference equations for $\psi$. These equations are solved by a block iterative, successive-overrelaxation method where adjustments are simultaneously made to an entire row of points in alternating directions on successive iterations. When the variation in one direction dominates, the number of iterations required to reach convergence is of the order of one.

The results of two calculations of the post-implosion phase of the Scyllac full torus experiment are described. These results are in agreement with theory, where comparison with theory is possible, but they indicate that the degree of toroidal force balance achieved with a particular coil configuration depends upon the plasma profile.

The initial conditions for the two calculations differ only in the plasma profile interior to the plasma vacuum interface. In both, the radius of the interface is 1 cm, and the average radius of the coils is 8 cm. The correspondence between these parameters and the others characterizing the Scyllac full torus experiment are close. In the first calculation, the plasma pressure is constant, and in the second, the plasma pressure varies as in the rigid rotor profile, with a characteristic radius equal to 0.4556. For this radius, the density at the interface is 10$^{-4}$ its value at the center of the column. Since the plasma $\beta$ at the axis is the same in both cases, the mass per unit length of the plasma column in the two calculations is 20% of its value in the sharp boundary case. The computation mesh is bounded by the interior of the shaped field coil as shown by plan and side views of the mesh in Fig. 1. Different shaped coil designs are modeled simply by moving the outer surface of the mesh. The computational domain encompasses one helical period of the Scyllac torus. Periodic boundary conditions are imposed at the ends of the sector. (Computing only a single helical period suppresses all but infinite wavelength instabilities while retaining the physics of the toroidal geometry.) The computation mesh has 10 zones in the toroidal direction, 20 zones in the poloidal direction, and 15 zones in the radial direction. The mesh has enough resolution to identify the significant features of the flow.

Initially, the plasma is perfectly axisymmetric and situated at the center of the discharge tube as it is observed to be immediately after implosion. In response to the helical fields produced by the shaped coils, the plasma surface distorts into a bumpy, helical shape. The development of the $\ell = 0$ and $\ell = 1$ perturbations of the surface is indicated in Figs. 2 and 3 respectively. In Fig. 2, the oscillation of the $\ell = 0$ perturbation of the interface in the sharp boundary case due to axial pressure equilibration is represented by the upper curves. The period for this oscillation is 2.3 usec. The corresponding perturbation of the plasma vacuum interface in the diffuse case should not be compared directly with theory, which gives the value of $\delta$ at the magnetic axis. However, an independent, two-dimensional calculation of the $\ell = 0$ perturbation of a cylindrical plasma is in agreement with the three-dimensional result. (In future calculations, trace particles will be used to mark interior flux surfaces, and direct comparison with theory will be possible.)

In Fig. 3, the helical displacement of the plasma from the magnetic axis is...
plotted. The upper curve represents the sharp boundary result. Again, the response is a damped oscillation about the design value with a period, equal to 1.04 μsec, which is shorter than for the \( \xi = 0 \) response. The lower curve represents the response of the diffuse plasma, which oscillates aperiodically. Its final value, when normalized to correspond to the theoretical definition, is 1.16 as compared with theory, which gives a value 1.14 for these parameters. The aperiodicity of the oscillation is a reflection of the dispersive character of the diffuse profile.

A visual comparison of the results for the two cases is afforded by Fig. 4, where the plasma surface is depicted at 1 μsec for each of the two profiles. The apparent distortion of the surface in the sharp boundary case is due to the non-circular cross-section of the plasma.

The approach of the plasma to toroidal force balance is indicated in Fig. 5, where the variation in time of the average outward radial velocity of the plasma is plotted. (The symmetry of the geometry implies that the average vertical velocity is zero. Computationally, the vertical velocity is essentially zero.) Here the toroidal drift force not counterbalanced by the \( P_a \), force due to the helical, bumpy field, the acceleration would be approximately 5.0 cm μsec\(^{-2}\). In the sharp boundary case, the acceleration is about 1.0 cm μsec\(^{-2}\), and in the diffuse case, the acceleration is approximately 2.7 cm μsec\(^{-2}\). The conclusion these results support is that the conditions for which toroidal force balance is nearly achieved in the sharp boundary case do not result in force balance in the diffuse case.

These results represent the beginning of an extensive numerical investigation of questions related to equilibrium in the Scyllac experiment. Even so, the numerical calculations have already demonstrated that the parameters for which toroidal force balance is achieved depend on the plasma profile. The goal of future calculations is to demonstrate a set of parameters for which toroidal force balance is exact, and thus to define a more favorable shaped coil design for future experiments.

**REFERENCES**


**LIST OF FIGURES**

1. Plan and side views are shown of the surface of computation mesh lying on the inner surface of the shaped compression coils.
2. The development of the \( \xi = 0 \) perturbation of the plasma vacuum interface for diffuse and sharp boundary cases is plotted.
3. The development of the \( \xi = 1 \) helical displacement of the plasma axis is plotted for the diffuse and sharp boundary cases.
4. The average radial velocity of the plasma for the diffuse and sharp boundary cases is plotted against time.
5. The plasma surface is depicted at 1 μsec for the sharp boundary case on the left, and the diffuse profile on the right.
Numerical Studies of Resistive Instabilities

D. Biskamp and R. Miesch

In the context of resistive instabilities which lead to hydromagnetic turbulence, a model is presented which describes the growth of instabilities in the solar atmosphere or magnetic oscillations in tokamaks. This model is applied to the case of a resistive plasma and is used to study the growth of instabilities in the solar atmosphere and in tokamaks.

The model is based on the equations of resistive magnetohydrodynamics, which are solved numerically using a finite-difference method. The results show that the growth of instabilities is strongly affected by the resistivity of the plasma.

In the case of the solar atmosphere, the model predicts that the growth of instabilities is strongly influenced by the magnetic field topology and the solar wind. The results are compared with observations of solar activity and are in good agreement with the data.

In the case of tokamaks, the model predicts that the growth of instabilities is strongly influenced by the magnetic field configuration and the plasma parameters. The results are used to optimize the tokamak design for plasma confinement.

The model also shows that the growth of instabilities is strongly affected by the presence of tearing modes, which are found to be a common feature of resistive magnetohydrodynamics.

The model is a valuable tool for understanding the growth of instabilities in both laboratory and astrophysical plasmas and has important implications for the design of tokamaks and the understanding of solar activity.
Both the nonlinear and the linearized scheme have been used, with no noticeable difference. Nevertheless the linearized scheme was preferred since we feel it to be numerically safer, no multiple solutions being possible. The main advantages of the implicit scheme appears to be that it is more quiescent than the explicit scheme, a smaller smoothing term (or none at all) being necessary.

Most of the results obtained to date refer to the tearing instability in a plane current sheath. The theory of the tearing instability valid for \( \eta \ll 1 \) predicts growth rates \( \gamma = \eta^{1/2} \) [1], saturation of the exponential growth at \( \psi = \eta^{1/2} \) and subsequent linear growth of magnetic islands \( \psi^{1/2} = \eta \tau \) [2]. We recover the theoretical scaling of \( \gamma \) for \( \eta < 10^{-3} \) and that of \( \psi \) for \( \eta < 10^{-4} \). We find that \( \psi \) is rather sensitive to any type of viscosity which strongly affects the build up of the associated convective flows producing the nonlinear currents. The subsequent phase of linear growth of magnetic island, \( \psi^{1/2} = \eta \tau \), becomes apparent only for \( \eta < 10^{-3} \).

If modes with different wave number \( k_y \) are allowed in the system, the mode with the largest wave length dominates nonlinearly. Starting with a shorter wave length perturbation, the corresponding magnetic islands tend to coalesce at a certain amplitude, unless only one island is left. Such effect is well known for collisionless tearing modes [4], [5]. It explains that in tokamaks only the \( n = 1 \) modes are observed.

The \( \eta_0 \)-profile and the magnitude of the external fields \( E_0 \) strongly affect on the nonlinear behavior. For a resistive equilibrium \( \eta_0(x) j_0(x) = \text{const} \) the modes grow to large amplitudes, somewhat faster for \( E_0 < \eta_0 j_0 \) and somewhat slower for \( E_0 > \eta_0 j_0 \). For \( \eta_0 = \text{const} \), however, saturation occurs at very low amplitude. Only if the back-ground variation is suppressed by artificially assuming a spatially varying \( E_0 \), \( E_0 = j_0 \), growth to larger amplitudes is recovered.

When the plasma is forced to flow with some average drift along \( B_0 \), tearing modes are growing in the moving frame. If, however, a stationary magnetic perturbation of sufficient strength is applied, the magnetic islands of the tearing mode become stationary, too. This may be related to recent experimental observations of suppression of Mironov oscillations in a tokamak by a small applied helical magnetic field [6].

Acknowledgements: The authors are grateful to M. Walter for her very valuable programming support.

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"This work was performed under the terms of the agreement on association between the Max-Planck-Institut für Plasmaphysik and EURATOM".
Numerical Solutions of the Twodimensional, Nonstationary, Nonlinear, Compressible, Resistive MHD Equations for Astrophysical and Laboratory Applications.†

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There are numerous theories and qualitative considerations concerning the mechanism of solar flares. It is still uncertain which of the theories are valid and if MHD theory allows for the sudden energy dissipation observed. To study these and other problems we are interested in a numerical solution of the non-ideal MHD equations in two dimensions such that the magnetic field lies in the $x$-$y$-plane and the current flows parallel to the $z$-axis. We consider sound and Alfvén waves and allow dissipation due to finite conductivity. The equations in question are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \frac{1}{c_s^2} j \times \mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{4\pi} \mathbf{v}^2 \mathbf{B},$$

$$j = \frac{c_s^2}{4\pi} \mathbf{v} \times \mathbf{B}, \quad \mathbf{v} \cdot \mathbf{B} = 0.$$

$\eta$ is the conductivity. As equation of state we take $p = c_s^2 \rho$ with the sound speed $c_s^2 = \text{const}$ throughout the rectangle considered. We make system (1) dimensionless, introduce the magnetic potential $A(x,y)$ such that $B = 2a_0 B_A$ and $\sigma = \ln p$. With the magnetic Reynolds number $R_M = 4\pi \sigma_0 V_A^2 / c_s^2 \eta$ where $V_A$ is a typical Alfvén speed, $\sigma_0$ is a macroscopic scale length and $a = c_s / V_A$, one can write system (1) as

$$\frac{3}{\Delta t} \frac{\partial f}{\partial t} + \nabla f = 0,$$

$$\frac{3}{\Delta t} \frac{\partial g}{\partial t} + \mathbf{v} \cdot \nabla g + \alpha^2 \nabla^2 g + \exp(-\sigma) \nabla^2 \psi \nabla \psi = 0,$$

$$\frac{3}{\Delta t} \frac{\partial \psi}{\partial t} + \nabla \psi + \nabla \chi = -R_M^{-1} \nabla^2 \psi = 0.$$
accuracy in $\Delta t$, if $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ commute, i.e. if $\frac{\partial}{\partial t} \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \frac{\partial}{\partial t}$.

This is accomplished for example if i) the derivatives with respect to $x$ and $y$ are distributed to $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ as mentioned, ii) if the nonlinear coefficients are held constant over one time step. This principle can be applied a second time to $\frac{\partial}{\partial t}$ such that $\frac{\partial}{\partial t} \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \frac{\partial}{\partial t}$ with $\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x}$. Clearly $\frac{\partial}{\partial t} \frac{\partial}{\partial x} = 0$ and we obtain

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} v_x + \frac{\partial}{\partial x} v_x &= 0, \\
\frac{\partial}{\partial t} v_y + \frac{\partial}{\partial x} v_y &= 0, \\
\frac{\partial}{\partial t} \sigma + \frac{\partial}{\partial x} \sigma &= 0, \\
\frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x} \psi &= 0,
\end{aligned}
\end{equation}

(5)

and

\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} v_x + \alpha^2 \frac{\partial}{\partial x} \sigma + e^{-\alpha} \frac{\partial}{\partial x} \psi &= 0, \\
\frac{\partial}{\partial t} v_y + e^{-\alpha} \frac{\partial}{\partial x} \psi &= 0, \\
\frac{\partial}{\partial t} \sigma + \frac{\partial}{\partial x} \sigma &= 0, \\
\frac{\partial}{\partial t} \psi - R^2 \frac{\partial}{\partial x} \psi &= 0,
\end{aligned}
\end{equation}

(6)

System (5) are one-dimensional hyperbolic equations of first order of the simplest kind. Provisions can be taken to build in flux-corrected transport or other means to adequately represent a developing shock. In system (6) the last equation is solved implicitly, which makes the solution of the second equation trivial. The first and third equation can be transformed into two linear hyperbolic equations of first order for one variable only, by introducing the Riemann invariants ($v_x + \alpha \sigma$). Thus we have split up system (1) into simple one-dimensional equations. In complete analogous fashion the integration of eq.(4b) is accomplished and one half cycle is completed.

We assumed that the values of the dependent variables at $t=(n+\frac{1}{2})$ were given. We redefine now the variables by $\sigma^{n+\frac{1}{2}} = (\frac{1}{2}) (\sigma^n + \sigma^{n+1})$ etc. and use these values and $\sigma^{n+1}$ etc. for the next cycle, which will produce values $\sigma^{n+\frac{1}{2}}$, and so on. It is easy to see that this procedure i) is of second order ii) closely links the integer and half integer time levels so that there is no danger of them running apart. The boundary conditions can easily be adjusted to the problem considered.
A comet provides an extended spherical-symmetric source of plasma in the interplanetary medium. This additional plasma causes a bow shock, brakes the onstreaming hypersonic solar wind plasma further, bends and amplifies the magnetic fields imbedded in the wind and occupies exclusively a field-free cavity extending downwind from the cometary nucleus. The parameters of cometary orbits and hydromagnetic wave propagation in the cometary plasma are such that for a stationary solar wind a stationary solution in a frame of reference resting in the cometary nucleus is to be expected. The plasma flow in the outer braking region behind the bow shock is rotationally symmetric and nearly incompressible. It therefore amplifies the transverse field-strength proportional to the inverse square root of the flow velocity till the magnetic stresses compete with the gas pressure. In the inner braking region these stresses produce asymmetry in the flow and amplify the total pressure on the innermost cavity above the gasdynamic stagnation pressure. To calculate the unknown 3-dimensional flow in this region which is likely to produce the conspicuous structures in the plasma tail of fluorescent cometary CO, we integrate the time dependent hyperbolic MHD-equations for negligible resistivity by finite differences in a suitable Eulerian grid of small volume elements. We write the equations as conservation laws in integral form (with respect to volume \( V \) and surface \( F \) of the grid cells)

for the conservation of mass

\[
\frac{3}{3t} \int_V \rho dV + \int_F \rho \left( \vec{u} \cdot d\vec{F} \right) = \int_V A \vec{d}V,
\]

of momentum

\[
\frac{3}{3t} \int_V \rho \vec{u} dV + \int_F \rho \vec{u} \left( \vec{u} \cdot d\vec{F} \right) + \left( p + \frac{B^2}{2} \right) d\vec{F} - \frac{1}{3t} \int_F \vec{B} \left( \vec{d} \vec{F} \right) = \int_V \lambda \vec{d}V
\]

of energy

\[
\frac{3}{3t} \int_V \rho g dV + \int_F \rho \left( g \vec{dF} \right) = \int_V A \frac{\rho}{2} dV
\]

with energy density

\[ g = \frac{B}{V^2} + \frac{1}{2} \rho u^2 + \frac{\rho}{8} \]

and energy flux density

\[ q = \left( \frac{B}{V^2} + \frac{1}{2} \rho u^2 + \frac{B^2}{4} \right) u - \frac{1}{4} \vec{u} \cdot \vec{B} \]

and of magnetic field

\[
\frac{3}{3t} \int_V \vec{B} dV + \int_F \vec{B} \left( \vec{u} \vec{dF} \right) - \int_F \vec{B} \left( \vec{d} \vec{F} \right) = 0.
\]

The source terms on the r.h.s. of the first three equations represent the plasma production by (photo-) ionization of neutral gases evaporating from the cometary nucleus. These terms are roughly proportional to the inverse square of the distance from the cometary nucleus. A method suitable for the construction of stationary solutions has to cope with a bow shock, a range of \(-10^5\) for the distance from the nucleus between bow shock and inner cavity, a range of \(-10^6\) in density, of \(-10^4\) in the source term, and of \(-10^2\) in the magnetic stresses, and with a strongly bended grid. We apply the 1. order-method of Godunov e.a. (1961) which makes use of a flexible grid, of the characteristics to treat the discontinuities at the cell boundaries, and of locally adjusted time steps. To avoid a free surface problem nearly irrelevant to our task, we assume a pencil-shaped rigid inner boundary near to the expected location of the fieldfree cavity. In a first step we produce appropriate gas dynamic boundary values on an outer surface immediately behind the shock and suitable initial values throughout the region from a rotationally symmetric code (Brosowski and Wegmann 1972) describing the flow for negligible magnetic stresses.

In a second step asymptotic cometary magnetic field data for low 1/8, i.e. for negligible Lorentz forces are constructed from this rotationally symmetric hydrodynamic flow by integrating the transport equation along streamlines, where it can be written as an ordinary equation

\[
\frac{3}{3t} + \left( \vec{u} \cdot \vec{dF} \right) \vec{B} = \frac{d}{dt} \vec{B} \quad \text{(upper case)}
\]

Fig. 1 gives the results for a transverse field \( \vec{B}_0 = 5.10^{-5} \) Gauss in the unperturbed solar wind. The maps show the streamlines, the fieldlines and the values of 1/8 in the central planes containing \( \vec{B}_0 \) and perpendicular to \( \vec{B}_0 \). The amplification and the anisotropy in 1/8 are obvious and this result encourages further study of the flow including the influence of Lorentz forces which should allow us to predict the appearance of visible field-aligned cometary plasma condensations from a given history of tangential discontinuities in the onstreaming solar wind. The latter appear with an average frequency of -1 per hour and
consist of sudden rotations of $\vec{B}_o$ by $30^\circ$ or more. It is improbable
that the nearly hourly appearance of the thin cometary plasma streamers
almost depicting the computed fieldlines of fig. 1a and the same fre-
quency in the solar wind discontinuities is a mere coincidence (Schmidt
1974). Rather it may be expected that selective effects of the Lorentz
forces quantitatively produce the observable aligned density enhance-
ment on the central flux tubes crossing the axis which are exposed to
the cometary plasma production term $A$ for the longest time interval and
which cannot be accelerated as effectively as all the other flux ele-
ments off axis. The field values constructed by integration along in-
terpolated streamlines are not suitable for the construction of the
Lorentz forces as this would amount to a second numerical differentia-
tion within the new grid of streamlines and fieldlines. Instead the
next step of our study was the separate but simultaneous treatment of
a) the "hydrodynamic" system of conservation of mass, momentum and
energy with its characteristics for a given field $\vec{B}$ and b) the field
transport equation with its characteristics for given velocity $\vec{U}$. As
outer boundary values we take the results of the first two steps for a
surface immediately behind the bow shock, where the magnetic stresses
can still safely be neglected for realistic parameters of solar wind
and comet. But since the transport equation of the field $\frac{\partial \vec{B}}{\partial t} = \text{curl}(\vec{U} \times \vec{B})$
describes the change of the field as a curl it has only two character-
istics running along the streamlines of the flow whereas the divergence
of the field is locally conserved, which in fact produces an Eigen-
wert 0 in the characteristic matrix. Unavoidable errors of numerical
approximation therefore produce growing divergences in the field. As a
remedy we changed the transport equation such that it transports not
only the divergence-free part of $\vec{B}$ but also div $\vec{B}$ itself with the flow
velocity $\vec{U}$ by adding a term $\frac{\vec{U}}{\rho} \times (\vec{B} \times \vec{U})$ in the above equation of conservation
of magnetic field. Though the results are qualitatively similar to the
results of the first two steps the divergences are not suffi-
ciently destroyed by the convection and in the "hydrodynamic" part
there arise longitudinal oscillations which are the magnetic analogue
of plasma oscillations arising from electric divergences. Therefore
the task is now to find a method of damping locally the artificial
div $\vec{B}$ without any change to the stationary solution.

B. Brosowski, R. Wegmann, MPI-PÄ/PA Astro 46 (1972).

Fig. 1 maps of magnetic data in rotationally symmetric flow around a comet.

a) map of stationary streamline and coplanar field-
lines for transverse field before shock; traveltime be-
tween subsequent fieldlines: $\Delta t = 1$ h.
b) map of $\frac{B^2}{8\pi}$ in the central plane
containing $\vec{B}_o$, initial field strength:
$B_o = 5 \times 10^{-5}$ Gauss.
c) map of $\frac{B^2}{8\pi}$ in the central plane per-
pendicular to $\vec{B}_o$, initial field
strength: $B_o = 5 \times 10^{-5}$ Gauss.
Numerical Study of Anisotropic Magnetohydrodynamic Turbulence

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Cosmic rays streaming through a background plasma generate Alfvén waves, which propagate in a "forward" direction with respect to the local ambient magnetic field \( B \). Provided the streaming speed is greater than the Alfvén speed \( V_A \), this wave-particle interaction results in a reduced streaming speed, near isotropy, and increased lifetime of the particles in the galaxy\(^\text{[7,12]}\), consistent with observations. In addition, this produces an asymmetric input to the wave spectrum at resonance with the most numerous cosmic rays, while turbulent interactions widen the spectrum and hence lead to resonances with less abundant cosmic rays.

Dimensional arguments predict a spectrum of weak MHD turbulence decaying towards large k as \( \propto k^{-3/2} \) (constant energy flux to higher k) where \( \int \xi_k dk = \text{total wave energy density} \). Another steady state solution has equipartition of energy among wave modes \( \xi_k \) with no net energy flow between modes. This gives a spectrum as shown in figure 1, although any k-dependent anisotropy requires more detailed analysis.

The full time evolution of a spectrum of MHD waves can be formulated in terms of wave-wave interactions\(^\text{[5]}\). First we consider only interactions among Alfvén (A) and fast magnetosonic (X) waves in a \( V_A < v < c \) plasma (where \( V_A \) is sound speed) as these modes carry \( 0 < v_\perp < c \), so the energy for a given k as the slow magnetosonic (S) waves. For reasonable spectra, only interactions among waves of comparable \( k \) contribute significantly to the time evolution of the spectrum.

Accordingly, we divide frequency-space into octaves each having \( \Delta \omega \). To study the anisotropic behaviour of the spectrum, we further divide each octave \( n \) into several propagation directions \( \mu = \cos \theta \) and combine all wave states within \( \mu \) about \( \mu \) at octave \( n \) into one state for each wave mode \( \sigma \) \( (\sigma = A, X) \).

Averaging the full time evolution equation over these domains for an interaction
\[
\frac{dN_{\sigma \mu \nu}}{dt} = \frac{dN_{\sigma \mu \nu}}{dt} - \frac{dN_{\sigma \mu \nu}}{dt} + \frac{dN_{\sigma \mu \nu}}{dt}
\]

which automatically conserves energy \( \int \xi \) by virtue of the octave approximation we find

\[
\frac{dN_{\sigma \mu \nu}}{dt} = \frac{dN_{\sigma \mu \nu}}{dt} - \frac{dN_{\sigma \mu \nu}}{dt} + \frac{dN_{\sigma \mu \nu}}{dt}
\]

where \( \frac{dN_{\sigma \mu \nu}}{dt} \) is the number density of plasmons of type \( \sigma \) in the domain \( \mu \) at octave \( n \) so that \( \sum_{\sigma, \mu, \nu} \frac{dN_{\sigma \mu \nu}}{dt} = \text{total wave energy density} \).

\( V_\sigma \) is the dimensionless k-volume of the domain \( (\sigma, \mu, \nu) \); and \( a \) is a numerical coefficient representing the strength of the particular interaction (1). The time evolution of the \( N_{\sigma \mu \nu} \)'s is then found by

\[
\frac{dN_{\sigma \mu \nu}}{dt} = \left( \text{source} \right)_{\sigma \mu \nu} + \sum_{\text{all allowed interactions with}} \left( \frac{dN_{\sigma \mu \nu}}{dt} \right)_{\text{each interaction from (2)}}
\]

Equation (3) was integrated in time numerically using Gear's method\(^\text{[2]}\).

For an asymmetric source, no steady state exists when only interactions among A and M waves are considered. All interactions in which an \( A \) wave would be the upper state cannot satisfy both energy and momentum (\( Mk \)) conservation, i.e. the "selection rules," (apart from \( A + A \rightarrow A \) which has a zero interaction strength) and therefore the energy and momentum fed into \( A \) waves can only be transferred to different \( k \) via interactions with the M waves. But \( A \) waves carry a z-momentum to energy ratio of \( \frac{Mk}{k} = \frac{V_A^2}{k} \), which means that the only input of \( A \) waves to transmit its z-momentum to energy ratio to different \( k \) and hence impossible to reach a steady state. Numerical experiments agree completely with the above analysis\(^\text{[6]}\).

Interactions with slow magnetosonic or "sound" waves (S) have been studied by various authors\(^\text{[1,6,8]}\) as a method for removing Alfvén waves when linear damping mechanisms are insufficient to balance any growth. The selection rules allow the transitions depicted in figure 2 which represent a constant flux of plasmons to lower \( k \). From the form of (2) we see that this has \( N_{\sigma \mu \nu} \propto (k)^{-1} \), To include S waves in our scheme, we choose \( V_A = V_B \) which makes the...
separation of states in figure 2 one octave and also adequately approximates realistic parameter values in a large fraction of interstellar space. The numerical results, displayed in figures 3 and 4, confirm the importance of the S waves in producing a steady state and dominating the spectrum to lower k. Other work has applied the sound cascade with some success to the problem of cosmic ray confinement.

I wish to thank Dr. J. Skilling for many fruitful discussions.

References

Figure captions
Figure 1 General MHD Spectrum
Figure 2 Transitions in the Sound Cascade
Figures 3 and 4 Steady state spectra for Alfvén (3) and fast magnetosonic (4) waves for all interactions among A, M, and S waves with infinite sound damping. Solid lines show theoretical slopes $\epsilon^2(k) \propto k^{-1}$ and $k^{3/2}$, $\epsilon^2(k) \propto k^2$ and $k^{-3/2}$. 

Figure 1

Figure 2

Figure 3
Expansion of Coronal Magnetic Fields in the Solar Wind
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Abstract. Some techniques and preliminary results will be presented concerning a two-dimensional numerical MHD-study of coronal magnetic field expansion.

Outline of the problem.
The problem of expansion of coronal magnetic fields under influence of the solar wind is reduced to two questions:
1. Behaviour of the level where the solar wind has a velocity \( > V_s \) or where the thermal pressure of the gas, including the thermalized decelerated solar wind \( > F_{mag} \), and
2. Estimate of the role of electrical conductivity in the life of a magnetic neutral sheet or a current sheet, the thickness of this sheet determined by the convective evolution, during the outbreak of a coronal magnetic field of the order of 100 Gauss.

Earlier attempts studied the distortion of a dipole field, when the solar wind is switched on. As a restriction, the field lines are taken perpendicular to the surface. A second attempt is the switch on of a solar wind, which forms after dictating its velocity at the solar surface. The next stage is, remembering \( V_{sw} \ll V_s \) at the surface, to modify the initial velocity of the switched-on "fast", such that \( V/\beta \) at every point, taking for \( \beta \) a dipole field with its center in the solar center. It can be shown, that a severe distortion of the magnetic field occurs if \( V_{sw} > V_s \). At a certain height, where \( V_{sw} \geq V_s \), the solar wind is still decelerated, which leads to a density increase, and a decrease of the Alfvén velocity, such that after some time, typically one week, \( p_{\text{sw}}\beta > p_{\text{sw}} \). Then a disturbance starts rising from the weakest point on the field line. If this disturbance has a large thickness, diffusion will not play a role, and the field lines will be transported outward until they break due to field rarefaction.

Auxiliary techniques.
For the large scale evolution, one is interested in the space around the Sun. From \( r = 1 R_\odot \) to \( 200 R_\odot \). An inverse-spherical transform is used, such that the usual spherical coordinates \((r, \theta, \phi)\) are mapped into \((r, \theta, \phi) \rightarrow (r, \theta, \phi)\), \( \rho = R_\odot/r, \mu = \cos \theta, \phi \).

This transform uses the geometrical tensor components

\[ h_1 = -R_\odot/\rho, \quad h_2 = R_\odot/(1 - \mu^2), \quad h_4 = R_\odot/\rho \]

with help of which the operations grad, div and curl can be defined after Landau-Lifshitz, Theoretical Physics, Appendix to Volume VIII.

In this geometry, for conservative schemes, one should use the convection velocities

\[ J_\nu = -V_{sw} R_\odot^2/\rho \]

and \( J_\epsilon = V_{sw} R_\odot \sin \theta/\rho \).

The program.
This is a modification of a 2-dim MHD-code, developed by J.P. Christiansen (Culham) on base of the GSHASTA/FCT-1 difference scheme (Bozis and Book, J.Comput.Phys.,13,29). For the inverse spherical problems isothermal and diffusion-free assumptions have been used, until now. The present program can be obtained from present author.

Two dimensions can be used if \( V_{sw} = 0 \) and \( R_\odot = 0 \) at \( t = 0 \), and if the computation does not include a diffusion coefficient \( \nu \) smaller than

---

Solar wind velocity \( V \) and logarithm of densities \( \rho \) as a function of time in six stages. At \( t=0 \) a disturbance is sent into the stationary expanding atmosphere. Left axis: solar surface. Right axis: infinity. The disturbance is seen to cross the critical point \((\mu=0, \rho_0)\) and to move this.
10^3; the runs are restricted to three months, real plasma time, sufficient for coronal phenomena. Then \( \dot{\theta} / \dot{\Phi} = 0 \) and \( \dot{x}_4 / \dot{t} = 0 \).

Solar wind disturbance.
A 1-dim test has been run in \( w \)-space for an isothermal solar wind for \( r \in [1, \infty) \). The stability is perfect. Oscillations in velocity \( \dot{\nu} \) as derived from momentum \( \rho \dot{v} \) and density \( \rho \) are of sound character, and do not propagate through the critical point nor influence the solution of the directly convected quantities \( \rho \) and \( \rho \dot{v} \). In figure 1 (preceding page), the solar wind momentum is increased at the surface and it is seen how the shock like correction forces the wind to become denser. The passage through the critical point is perfect.

Field blow up.
Figure 2 (top left) shows the distortion of a weak dipole field in \( (w_t) \)-coordinates, where the solar wind is switched on in region 1; the gas flow is parallel to \( \eta \), until it is speeded up beyond \( v_\phi \) and the field is compressed into a shock. In region 2, the matter stagnates until \( p_{gas} > p_{mag} \), and a gaussian-like disturbance is formed with thickness smaller than 30000 km. Field tearing off is obtained when the field is rarefied too much.

Magnetic Kelvin-Helmholtz.
Results of the nonlinear growth stage of Kelvin-Helmholtz instability at the rim of a magnetic flux-tube, due to nonisothermal strong magnetic diffusion will be presented at the conference.

Injection of gas in a field.
At the top of a dipole field, gas is injected along the field lines with \( v = v_\phi \). The formation of a thick disturbance is shown in fig. 3 when, due to the gas stagnation, \( v_\phi \) decreases below \( v \). This problem is diffusion free and in linear coordinates. Full MHD-equations can be solved for this kind of problems, leading to insight in a scale of cosmical processes characterized by a just super-Alfvénic plasma flow, or a gas pressure equal to the magnetic pressure. Applications have been made to magnetospheric, coronal and extragalactic situations.

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Density profiles (bottom) and contours (top) of a flow injected into a magnetic region (bottom: at right; top: bottom, marked by B), which expands in a broad disturbance into the field-free region which is in pressure equilibrium. The initial signal is reflected at the axis. Numbers in the contour plots show the Alfvén time.
MUSCL, A NEW APPROACH TO NUMERICAL GAS DYNAMICS

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At Leiden Observatory, an original second-order technique for solving ideal compressible flow problems has been developed. The heart of the MUSCL code (Monotonic Upstream Scheme for Conservation Laws) is a one-dimensional Lagrangian scheme. It is based on the integral conservation laws and is dissipative, so that it can be used across shocks. The Lagrangian results can be remapped onto any moving space-grid; higher dimensions are covered through time-splitting.

The Lagrangian scheme

The Lagrangian scheme is patterned after Godunov's (1) method, which may be regarded as the first-order member of the MUSCL family. The one-dimensional gas is divided into slabs. First, the distributions of specific volume \( \mathbf{V} \), velocity \( \mathbf{u} \) and pressure \( p \), generated in the previous time-step, are approximated in each slab by linear functions. In this approximation, the slab integrals of \( \mathbf{V}, \mathbf{u} \) and specific total energy \( E \) are conserved. The resulting overall approximation is discontinuous at the slab interfaces.

Next, the behaviour in time of \( \mathbf{u} \) and \( p \) at each slab interface is determined with first-order accuracy. The values right after resolution of the discontinuity are obtained using the jump conditions; the first time derivatives can be expressed in terms of the gradients in the interacting slabs. We then know enough to update the slab integrals of \( \mathbf{V}, \mathbf{u} \) and \( E \). For instance, conservation of momentum in slab \( (x_{i-1}', x_i) \) requires that

\[
\int_{x_{i-1}'}^{x_i} \mathbf{u}(t', x) \, dx = \int_{x_{i-1}'}^{x_i} \mathbf{u}(t, x) \, dx + \frac{\Delta t}{\Delta x} \int_{x_{i-1}'}^{x_i} p(t, x) \, dt
\]

all terms at the right-hand side are known with sufficient accuracy.

Updating the gradients may or may not require additional information on the structure inside each slab, depending on the accuracy desired.

The size of \( \Delta t \) is restricted by the usual Courant condition.

Slab interaction

The interaction of the slabs may be calculated in the locally linearized approach. This means that the Lagrangian sound speed is taken to be constant in each slab, for the duration of the time-step. If the interaction is strong, nonlinear wave speeds in reality will take over the role of the sound speeds; adhering to the linear approach then leads to nonlinear oscillations and, possibly, instability.

The nonlinear wave speeds can be calculated in an iterative way, starting from the sound-speed values. The number of iterations may be made a function of the size of the jumps in \( \mathbf{u} \) and \( p \) at the interface. In calculating the wave speeds, the equation of state of the gas may be approximated by a polytropic law.

Gradients inside a slab

The quality of the scheme varies considerably with the way of determining the gradients inside the slabs. The extreme possibilities are finite differencing and least-square fitting. For instance, in the first case, the velocity range \( \Delta u_{i-1/2} \) in slab \( (x_{i-1}', x_i) \) is given by differencing the neighbouring slab averages:

\[
\Delta u_{i-1/2} = \frac{1}{2} (\mathbf{u}_{i+1/2} - \mathbf{u}_{i-1/2}).
\]

In the second case, we must determine the first moment of the distribution \( \mathbf{U}(t, x) \) resulting from the preceding slab interactions:

\[
\Delta u_{i-1/2} = \frac{1}{2} \int_{x_{i-1}'}^{x_i} \mathbf{U}(t, x) \cdot (x - x_{i-1/2}) \, dx.
\]

The formula for the first moment of \( \mathbf{U} \) is easily obtained in the locally linearized approximation. For heavily tormented slabs, the sound speeds occurring in the formula may be replaced by nonlinear wave speeds.

An important consequence of Eq. (3) is that \( \Delta \mathbf{u} \) becomes an independent quantity and has to be stored separately, in addition to \( \mathbf{U} \). For a two-dimensional calculation, this means an increase in storage space of about a factor 2 per mesh. This increase can be offset by increasing the mesh width a factor \( \sqrt{2} \). Since the truncation error of the scheme with (3) is at least a factor 9 smaller than with (2), the use of (3) remains attractive.

Accuracy

The convective scheme underlying the above Lagrangian method turns out to be the upstream-centered second-order difference scheme of Fromm (2), if the gradients are determined as in (2). The convective scheme arising from (3) has not been previously used and is discussed in Van Leer (3).

Figure 1 shows the dispersion (for Courant number \( C > 0 \)) and the dissipation (for \( C = 1 \)) in the Fromm scheme, the MUSCL scheme with least-square error and the Phoenical SHASTA scheme of Book, Boris and Hain (4). All three are second-order schemes with a dispersive leading term in their truncation error. The quality of the MUSCL scheme is obvious.

A full description of the MUSCL code will be given in Van Leer (5).
Monotonicity

If the gradient of (say) \(U\) changes appreciably across a slab, strict evaluation of an average gradient according to (2) or (3) will lead to numerical oscillations. To prevent these, \(\Delta u_{i+1/2}^1\) must be reduced such that the values of the linear function used in the slab, that is,

\[
u(t,x) = \bar{u}_{i+1/2} + \frac{\Delta u_{i+1/2}^1}{\Delta x} (x - x_{i+1/2}), \quad x_i < x < x_{i+1},
\]

(4) remain within the range spanned by the adjacent slab averages \(\bar{u}_{i-1/2}\) and \(\bar{u}_{i+1/2}\).

If \(\Delta u_{i+1/2}^1\) happens to be an extremum, \(\Delta u_{i+1/2}^1\) is reduced to zero in order not to accentuate the extremum.

This procedure exactly ensures monotonicity and positivity when used in a linear convection scheme, cf. Van Leer (3). Its effect in a nonlinear scheme is somewhat less certain, but still quite valuable. This may be judged from Figure 2, which shows the results of a Lagrangian shock-tube calculation. In this case, gradients were obtained by differencing interface values.

Further provisions

The Euler-remapping, time-splitting and multi-fluid provisions are being developed by P.R. Woodward. The performance of the Euler scheme for the shock-tube problem mentioned above is also shown in Figure 2.

Third-order scheme

A third-order MUSCL scheme is currently being developed. In the Lagrangian step, the approximation inside each slab is quadratic, while the overall approximation can be made continuous at the interfaces. The interface values of \(V, U\) and \(p\) need to be stored in addition to the slab averages. The absence of discontinuities simplifies the interaction formulae so much that, without source terms, the third-order scheme requires less computing time than the least-square second-order scheme.

The underlying third-order convection scheme is discussed by Van Leer (3).

References
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Figure 1. Dispersion and dissipation in three second-order convection schemes.

Top: ratio \(\omega\) of numerical and exact convection speed, versus number of meshes in a wave-length \(\ell\), for vanishing Courant number \(\sigma\).

Bottom: damping factor \(|\rho|\) per time-step, versus \(\ell/\Delta x\), for \(\sigma = 1\).

Figure 2. Lagrangian and Eulerian shock-tube calculations with MUSCL. Gradients obtained by differencing interface values. Initial discontinuity at Euler coordinate \(r = 0\). Pressure ratio \(10^6\), density ratio \(10^4\). Initial zoning: \(\Delta r = 3\) for \(r < 0\); \(\Delta r = 5\) (Euler) or 10 (Lagrange) for \(r > 0\). Courant number \(\approx 0.25\). In the Lagrange run, starting errors show up near the contact discontinuity (c.d.). There also is a starting error in the position of the c.d., although it has acquired the correct speed. In the Euler run, the c.d. still moves too fast. Around c.d. and shock, consecutive mesh values have been marked.
We consider an incompressible, inviscid, two-dimensional fluid. The set of equations governing its behaviour may be written

\[ \begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).
\end{align*} \]

We consider this problem in a bounded domain \( D \) with boundary \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \). The boundary conditions on the fluid are given by the no-slip condition on \( \Gamma_1 \) and \( \Gamma_2 \), and the traction conditions on \( \Gamma_3 \) and \( \Gamma_4 \).

The velocity field is expressed in terms of an eigenfunction basis:

\[ u = \sum_{n=1}^{N} \phi_n(x) \psi_n(y), \quad v = \sum_{n=1}^{N} \phi_n(x) \psi_n(y), \]

where \( \phi_n(x) \) and \( \psi_n(y) \) are eigenfunctions of the Laplace operator in \( x \) and \( y \) respectively. The eigenfunctions are orthogonal and normalized.

The eigenvalues \( \lambda_n \) are obtained by solving the eigenvalue problem:

\[ \begin{align*}
\Delta \phi_n(x) &= \lambda_n \phi_n(x), \\
\Delta \psi_n(y) &= \lambda_n \psi_n(y).
\end{align*} \]

The linear stability of the base flow is determined by the sign of the real part of the eigenvalues. If all eigenvalues are negative, the flow is stable. If at least one eigenvalue has a positive real part, the flow is unstable.

The flow is analyzed using the Orr-Sommerfeld equation, which is a linearized stability analysis. The Orr-Sommerfeld equation is:

\[ \begin{align*}
\frac{\partial \phi_n}{\partial t} + \left( \frac{\partial^2 \phi_n}{\partial x^2} + \frac{\partial^2 \phi_n}{\partial y^2} \right) &= -\lambda_n \phi_n, \\
\frac{\partial \psi_n}{\partial t} + \left( \frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2} \right) &= -\lambda_n \psi_n.
\end{align*} \]

The solutions to the Orr-Sommerfeld equation are sought in the form:

\[ \phi_n(x, y, t) = \phi_{n0}(x) \psi_n(y) e^{i \omega t}, \quad \psi_n(x, y, t) = \phi_{n0}(x) \psi_n(y) e^{i \omega t}, \]

where \( \phi_{n0}(x) \) and \( \psi_n(y) \) are the eigenfunctions, \( \omega \) is the frequency, and \( \lambda_n \) is the growth rate.

The growth rate \( \lambda_n \) is determined by the boundary conditions at \( x = 0 \) and \( y = 0 \). The boundary conditions are:

\[ \begin{align*}
\phi_n(0, y) &= 0, \\
\psi_n(x, 0) &= 0.
\end{align*} \]

The growth rate is given by the dispersion relation:

\[ \int_0^\infty \int_0^\infty \left( \phi_{n0}(x) \phi_n(x) + \psi_{n0}(y) \psi_n(y) \right) e^{i \omega t} dx dy = 0. \]

The eigenvalues are obtained by solving the dispersion relation.

The instability is determined by the sign of the real part of the growth rate. If the real part is positive, the flow is unstable. If the real part is negative, the flow is stable.
3.1 CHOLESKY’S METHOD FOR THE CALCULATION OF THE ROOTS

Because the determinant D is an even function of $\omega$, we can calculate the $N$ pairs of roots $\omega$ by a numerical method using a Newton Lagrange interpolation between alternatively positive and negative values of $D$. This is a rather precise but lengthy method (30" per point for $N = 10$, 2" for $N = 14$ on the ICL 4/70).

The second method considers the fact that the determinant has the special form $D = S - \omega B$ where $S$ is a real symmetric matrix and $B$ is a real diagonal one. We want to solve $DV = 0$ or $V = uB$. The Cholesky method is based on the fact that a regular symmetric matrix $S$, all principal minors of which are different from zero, can always be written $S = LL^T$ where $L$ is a lower triangular matrix with non-zero diagonal and $L^T$ its transpose. So we write

$\begin{align*}
S = \omega B & = LL^T \\
V & = \omega B
\end{align*}$

and with $L^T = U$ we get $L^{-1} B (L^{-1})^T = \omega = \omega^T$$^T$

the matrix $A = L^{-1} B (L^{-1})^T$ is symmetric and its eigenvalues give the $\omega_i$.

If $S$ is positive definite, $L$ and $L^T$ are real and $A$ is real symmetric, therefore the calculation of the $\omega_i$ is straightforward. If $S$ is not positive definite, $A$ is composed of real or purely imaginary elements (but is Hermitian in the general case) and we must use more elaborate routines. The calculations are much faster than with the numerical method (25" for $N = 14$).

3.2 RESULTS FOR THE PLANE SHEAR FLOW

The reduced roots $\omega_i / k$ are localized between fixed boundaries, the $U_i$. ($N - 1$) reduced eigenfrequencies are always real, located in the intervals $(U_1, U_2), (U_2, U_3), \ldots, (U_{N-1}, U_N)$, the left and right limits being obtained for $k = 0$ and $k = \infty$ respectively. The last root is real located between $0$ and $U_1$ for $k$ larger than a critical value $k_c$ and is purely imaginary located between $0$ and $iU_N$ for $k < k_c$. The instability appears through a "marginal adiabatic mode" when $\omega$ goes through zero. The root becomes imaginary at low $k$ and carries almost the whole excitation when $k$ goes to zero, and has a proper physical meaning whereas the other real ones correspond to a mere discretization of the axis of frequency and depend on the number of bags.

We have plotted on Figure 1 the variation of the growth rate $-\omega / k$ with the wave number $k$ for different $N$. We observe that the convergence is very fast for small $k$ till for $k$ giving the maximum of growth rate. For larger $k$, the convergence is not so good, and in fact rather slow for $k_c$, due to "model noise".

4. THE JET FLOW CASE

If we consider an even velocity profile approximating two arcs of parabola, the corresponding vorticity distribution is an antisymmetric NMB with $2(N+1)$ contours for $N$ bags (ty_1, and $y = 0$).

For $N$ bags, the determinant of order $2N + 1$ has now the following form (EX for $N = 3$):

$$\begin{align*}
\omega + 1 & = (1) (1) (2) (2) (3) (3) \\
1 + \omega & = \frac{ku_1}{A_1} \\
0 & = 0
\end{align*}$$

where $(ij)$ still stands for $exp - k |y |_{ij}$ and $A_0 = -2(A_1 + A_2 + A_3)$.

This time the determinant is no longer an even function of $\omega$ and no numerical interpolation can be applied. But it is still possible to use the Cholesky method.

We find $(2N + 1)$ roots, two of them form a complex conjugate pair the imaginary part of which goes to zero as $k$ goes to zero, a well-known result for the jet case.

For $k$ large enough, all the roots become real and reach the limits $U_i$ when $k = \infty$.

Studies of the growth rate show, like in the case of a shear flow, that the convergence is very fast at small $k$ till the $k$ corresponding to the maximum of growth rate (Figure 2). For larger $k$, the graphs present strong oscillations due to the model noise. A study is in progress at the moment to suppress or at least attenuate this model noise, and unless this is possible it will be hopeless to try to use this NMB model for non-linear or even slightly non-linear problems.

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EXACT MAGNETIC ENERGY CONSERVATION FOR A NON-UNIFORM MESH

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1. INTRODUCTION

Numerical methods for solving partial differential equations in physics normally have to resolve non-physical difficulties which arise from the use of finite difference schemes, e.g. numerical stability, dispersion, truncation. Richtmeyer and Morton [1] have surveyed numerical methods for solving a variety of linear and non-linear differential equations. These methods when applied to a single equation can guarantee numerical stability and at the same time enforce conservation of the quantity that is being integrated.

In magnetohydrodynamics it is in general necessary to solve a set of coupled partial differential equations in the variables \( \rho, \mathbf{v}, T \) and \( \mathbf{B} \), denoting density, velocity, temperature and magnetic field respectively. Although the MHD equations conserve mass, momentum, energy and magnetic flux these conservation properties are not necessarily retained in the finite difference formulation. Roberts and Potter [2] who studied various numerical methods for solving the MHD equations, pointed out that even if total energy is conserved it is nevertheless possible to produce errors in the individual contributions to the total energy. A small error in the magnetic energy

\[
\mathcal{W}_B = \int \frac{1}{\mu_0} \mathbf{B}^2 \, dv
\]

(1)
can for example produce a large error in the thermal energy

\[
\mathcal{W}_\text{th} = \frac{1}{2} \rho \left( \mathbf{v} \cdot \nabla \mathbf{v} \right) \, dv
\]

(2)

(p is thermal pressure) if \( \mathcal{W}_B \gg \mathcal{W}_\text{th} \). While it may be straightforward to conserve linear quantities such as

\[
\mathcal{W}_B = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \, dv
\]

(3)

there is no general guarantee that for quadratic or cubic quantities like

\[
\mathcal{W}_B = \frac{1}{2} \rho \left( \mathbf{v} \cdot \nabla \mathbf{v} \right) \, dv
\]

(4)
or \( \mathcal{W}_B \), one can find exact difference identities [1].

The approach pursued in this paper is to look for a non-linear identity which holds in particular cases. In sections 2 and 3 we study the magnetic field diffusion equation coupled to a temperature equation. To highlight the principles established we restrict the discussion without loss of generality to equations in one spatial dimension; cylindrical geometry \((r, \theta, z)\) is assumed. The effects of enforcing exact conservation of magnetic energy and thermal energy are illustrated by results from computer calculations.

2. COUPLED DIFFUSION EQUATIONS

The coupled magnetic field and temperature equations are

\[
\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times \mathbf{E}
\]

(5)

\[
\frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T) + J \cdot \mathbf{E}
\]

(6)

where \( \mathbf{E} = \nabla \times \mathbf{B} \) is the current density, \( \kappa \) the number density and \( \kappa \) the thermal conductivity. We have dropped additional source terms in (6) and only one temperature is introduced; there are however no restrictions imposed by these simplifications.

Conservation of total energy is demonstrated by multiplying (5) with \( \frac{1}{\mu_0} \mathbf{B} \) and adding it to (6). The resulting equation is integrated from \( r = 0 \) to \( r = a \) (wall or boundary). This gives

\[
\frac{d}{dt} \left[ \mathcal{W}_B + \mathcal{W}_\text{th} \right] = a \left( \kappa \frac{\partial T}{\partial a} + a \left( \mathbf{E} \times \mathbf{B} \right) \right).
\]

(7)

The finite difference algebra proceeds in a similar way and leads to exact conservation of \( \mathcal{W}_B + \mathcal{W}_\text{th} \) according to (7), provided the appropriate formula is used for the ohmic heating term \( J \cdot \mathbf{E} \). We obtain a perhaps complex, but nevertheless quite precise prescription for evaluating \( J \cdot \mathbf{E} \) and it is interesting to notice that \( \mathbf{E} \) can have any form as long as this identical form is used in both (5) and (6).

3. FINITE DIFFERENCE FORMS

A fixed non-uniform mesh of \( N \) intervals

\[
o = r_1, r_2, r_3, \ldots, r_N, r_{N+1} = a
\]

is used. The variables \( n, T, B \) and \( B \) are defined at the half-integral points

\[
r_{j+\frac{1}{2}} = \frac{1}{2} (r_j + r_{j+1})
\]

(9)

and are to be regarded as cell averages, e.g. \( (B_j \Delta A)_{j+\frac{1}{2}} \) and \( (\frac{\partial T}{\partial a})_{j+\frac{1}{2}} \) ame the flux and thermal energy of cell \( j+\frac{1}{2}, \Delta A \) is the volume per unit length in \( z \), i.e.

\[
\Delta A_{j+\frac{1}{2}} = \frac{1}{2} (r_{j+1}^2 - r_j^2)
\]

(10)

The finite difference forms of (5) and (6) are

\[
\frac{d}{dt} \left[ (B_j^2 - B_0^2)_{j+\frac{1}{2}} \right]_{j+\frac{1}{2}} = \Delta t \left( (E_j^2 - E_{j+1}^2) - (E_{j-1}^2 - E_j^2) \right)
\]

(11)

\[
(\Delta A (B_j^2 - B_0^2)_{j+\frac{1}{2}})_{j+\frac{1}{2}} = - \Delta t \left( (E_j^2 - E_{j+1}^2) - (E_{j-1}^2 - E_j^2) \right)
\]

(12)

\[
\left( \frac{\Delta A}{\gamma-1} \left( (T_j^{2+1} - T_j^2) \right) \right)_{j+\frac{1}{2}} = \Delta t \left( (\kappa \gamma \nabla T)_j - (\kappa \gamma \nabla T)_{j+1} + \Delta A (J_0 E_0 + J_0 E_0)_{j+\frac{1}{2}} \right)
\]

(13)
Exact conservation of energy has thus been established for a one-dimensional system of coupled diffusion equations. It is easy to see that if equations (5) and (6) are solved in two or three dimensions by either the ADI method (1) or a splitting method (1) then (11)-(13) still apply; $s_{j+b}$ given by (22) should then be calculated after each sweep in $(x,y,z)$ or $(r,s,z)$ of (5) and used in (6). Magnetic fields should hence be diffused before temperatures in order to retain the exact conservation of energy.

4. CALCULATIONS

We study the ohmic heating of a plasma in a reversed field pinch configuration by solving equations (5) and (6) as described. Figure 1 shows the temporal evolution of the temperature from 3 calculations. In the first calculation $E,J$ is as given by (22). The second and third calculations use the explicit form

$$E = -\frac{a}{b} v \cdot \nabla T$$

with $a = \frac{\mu_0}{\rho}$ and $b = \frac{\mu_0}{\rho}$. After 500 steps with constant $\Delta t$ we calculate the relative error $\epsilon$ in $W = W_B + W_A$,

$$\epsilon = \frac{W(t) - W(0)}{W(t)}$$

where the total energy input $W_{in}$ is obtained from (7).

$$W_{in} = \int_0^{500} \Delta t \frac{a}{b} \nabla \cdot \left( \frac{\mu_0}{\rho} \nabla T \right)$$

We find for the three cases: $\epsilon_1 = 9.8 \times 10^{-18}$ (round off), $\epsilon_2 = 0.0027$ and $\epsilon_3 = 0.028$. The errors $\epsilon_1$ and $\epsilon_3$ may not seem large because $\frac{\Delta t}{a} >> \frac{\Delta t}{b}$.

It therefore seems desirable to retain the exact conservation of energy by using the expression presented; this improves the confidence in the results obtained.

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"The influence of aliasing errors on nonlinear wave propagation"

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It is of great interest in plasma physics to compute the time history of finite amplitude waves, since effective heating may be associated with wave instabilities. If interactions in more than one space dimension are considered one needs an effective numerical scheme working well in the régime of rather low cut-offs.

We have tested two methods by integrating nonlinear scalar wave equations, and by comparing the results with analytic solutions. Both methods consist of integrating equations for time derivatives of Fourier coefficients by a linearly stable predictor-corrector scheme of fourth order; therefore the linear parts of the wave equation are equally well represented by both methods. The difference lies in the nonlinear terms which differ due to the aliasing interactions. These appear when local products, defined on a grid in x-space, say \( u(x_j)v(x_j) \), are transformed into Fourier space. For method 1 ("spectral method") we use shifted grids in x-space in order to eliminate aliasing terms (Patterson & Orszag, 1974). For method 2 ("pseudospectral method") we use simply the Fourier transform of the local product \( u(x)v(x) \).

In fig. 1 \( \varphi_{\text{analyt.}} \) is the well-known solitary wave solution of the Korteweg-de Vries equation. The dotted line, \( \varphi_{\text{num 1}} \), is the corresponding solution obtained after 1,000 time steps, according to method 1; it coincides with \( \varphi_{\text{analyt.}} \) very well. The dashed curve, \( \varphi_{\text{num 2}} \), is obtained from method 2 under the same initial conditions; here the aliasing interactions lead to a decay of the soliton into a smaller one plus a dispersive wave packet, i.e., to a weakening of the nonlinearity. Fig. 2 and 3 show the corresponding spectra at 10 equidistant times according to methods 1 and 2, respectively. While in the former case the spectrum remains constant over a large range of wave numbers - as it should - we see in the latter case an overall change of the spectrum, propagating from large to small wave numbers. Method 1 turns out to be more effective; this has been confirmed by a run with method 1 and with half of the number of modes (\( N = 32 \)), i.e., the same computer time (and half of the storage) as required by method 2 with \( N = 64 \). The result is shown in x-space by the crosses in fig. 1, and in Fourier space by the spectrum of fig. 4. Similar results have also been obtained by using other model equations (Schamel & Elsässer, 1976). Therefore we conclude that the spectral method is more appropriate for wave propagation problems, in contrast to the conclusions of Fox and Orszag (1973) for incompressible 2-d Navier-Stokes turbulence.

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DEQTRAN - A PROGRAM GENERATOR FOR THE SOLUTION OF 1D SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: An implicit difference method for 1D systems of coupled, highly non-linear partial differential equations is described. An executable FORTRAN program based on this method can be generated completely automatically by DEQTRAN (Differential EQuation TRANslator), a special purpose language and compiler.

1.) Introduction.

In plasma physics solutions of time dependent one-dimensional partial differential equations (e.g., fluid and Maxwell’s equations) have been successfully obtained for more than a decade by using implicit difference methods. The term “implicit scheme” has been used in two different senses:

a) single equations of a system are treated implicitly, each with respect to one variable /1,2,3/;

b) simultaneously, the whole system and all variables in each equation are treated implicitly /4,5,6/.

This paper uses b) only.

Attempts to construct superprograms which contain all sorts of geometries, physical effects, number of fluids etc., as well as a special run would fail because such a program is necessarily unwieldy and always has to be kept abreast of current research. Such difficulties could be avoided by developing symbolic languages or standardized programming /7/. Here we present a new tool for fast and economic construction of efficient computer codes for such plasma models.

To this end, we have collected simple and widely applicable rules for constructing difference equations and their solutions (section 2). The application of these rules to any special system of equations is performed by the computer itself; i.e., the problem is formulated in a convenient form (DEQTRAN = Differential EQuation TRANslator, section 3) and the computer generates a complete FORTRAN program which may then be run in the usual way for given numerical data. In essence, this device may be considered as a “black box” into which the system of equations and the boundary conditions etc. are fed into which finally yields the numerical solution of the problem.

2.) Construction of Implicit Difference Schemes

In order to get from differentials to finite differences, we integrate over a cell \( \Delta x \), \( \Delta t \) (in any geometry)

\[
\int_0^1 \int_0^1 \frac{2}{\Delta x} U \, \Delta x \, \Delta t = \int_0^1 \int_0^1 (F) \, \Delta x \, \Delta t
\]

(1)

The operator \((F)\) may contain any combination of powers of components of \( U \) and spatial derivatives up to second order.

- 2 -

For evaluating the time integral on the right-hand side, we apply the mean value theorem and linearize all terms according to the rule

\[
\left\langle A^\alpha B^\beta \cdots G^\gamma \right\rangle = \frac{1}{2} \left\{ \left( A^\alpha \right)^{-1} B^\beta \cdots G^\gamma + \left( B^\beta \right)^{-1} A^\alpha \cdots G^\gamma + \cdots + \left( G^\gamma \right)^{-1} A^\alpha B^\beta \cdots + \left( A^\alpha + B^\beta + \cdots + G^\gamma - 2 \right) A^\alpha B^\beta \cdots \right\}
\]

(2)

With respect to the spatial dependence, we use the guide lines:

1. All variables are defined at the grid points.

2. Between the grid points, the variables are linearly interpolated.

These rules finally lead to a system of coupled linear equations of the following form:

\[
C_k^i \cdot U^k (i-1) + C_M_k^i \cdot U^k (i) + C_R_k^i \cdot U^k (i+1) + C_l^i = 0.
\]

(3)

Because the boundary conditions may contain not only combinations of the variables themselves but also their derivatives, we always put the boundary between grid points. The external variable values (at \( i=1 \) and \( i=N+1 \)) thus do not necessarily have a physical meaning. Then, the same difference construction rules as above lead for the left boundary to

\[
C_M_k^i \cdot U^k (i) + C_R_k^i \cdot U^k (i+1) + C_l^i = 0,
\]

(4)

and for the right boundary to

\[
C_R_k^i \cdot U^k (i-1) + C_M_k^i \cdot U^k (i) + C_l^i = 0.
\]

(5)

The solution of these linear systems might be obtained by using standard fast matrix inversion routines in connection with the efficient methods for tridiagonal matrices; however, it turns out that most physical models yield sparse matrices \( C_l, C_M, C_R \); hence, we use the Gaussian elimination which is a most economic procedure because all operations on zero elements can be left out from the beginning. All details concerning the difference approximation and the solution of the algebraic system are given in /8/.

3.) DEQTRAN (Differential EQuation TRANslator)

The procedure outlined above is very cumbersome even for moderately complicated models, not only in deriving the systems (3) ~ (5) but also in programming these steps as well as the optimized Gaussian elimination for use on a computer, not to speak of the considerable probability of errors. To overcome these difficulties, we have the computer perform the whole procedure. This was the purpose for which DEQTRAN was developed. DEQTRAN is a special purpose language which allows easy, computer-adapted formulation of the mathematical problem. This may be
illustrated by the following example:

\[
\frac{2}{\epsilon_t} \dot{Q} + \frac{2}{\epsilon_t} \left( \frac{\partial}{\partial x} (QV) \right) = 0,
\]
\[
\frac{2}{\epsilon_t} \dot{V} + \frac{2}{\epsilon_t} \frac{\partial^2}{\partial x^2} V + \frac{1}{\epsilon_t} \frac{\partial}{\partial x} (\dot{Q}T) = 0,
\]
\[
\frac{2}{\epsilon_t} \left( \frac{\partial}{\partial x} (QT) \right) + \frac{2}{\epsilon_t} \left( \frac{\partial}{\partial x} (QTV) \right) + (i - 1) \frac{\partial^2}{\partial x^2} V - \frac{2}{\epsilon_t} \left( \frac{\partial}{\partial x} (\dot{Q}T \frac{\partial}{\partial x} T) \right) = 0.
\]

This is written in DEQTRAN as follows:

\[
\text{DOT (RHO) + DIF (RHO K V) = 0}
\]
\[
\text{DOT (V) + 0.5 * DIF (V x V) + RHR x DIF (RHO x T) = 0}
\]
\[
\text{DOT (RHO x T) + DIF (RHO x T x V) + GAM1 x RHO x T x DIF (V)}
\]
\[
\text{ - DIF (KAP x T x T x DIF (T)) = 0;}
\]

All conventions and possibilities will be explained in detail in [9/]. With such input, DEQTRAN first constructs the difference schemes (3) - (5) and generates FORTRAN routines for their numerical evaluation. As a second step, the resolution procedure for (3) - (5) is performed and coded in FORTRAN. In order to produce a FORTRAN program ready for use, DEQTRAN completes the above steps by creating a main program and subroutines for accuracy checks, input of initial values and numerical and graphical outputs.

4.) Experience with DEQTRAN and Further Development

DEQTRAN has been in operation now for four years. It has been tested for systems of nonlinear diffusion equations, MHD equations, MFH equations /10/, etc. The present version of DEQTRAN can handle up to 20 equations and requires 240K bytes of storage; the CPU time for creating a FORTRAN program which solves a set of five equations is typically 1 - 2 seconds on an IBM/360-91. A few drawbacks can be found in our present version of DEQTRAN: products of derivatives cannot be processed; in eq. (2) only positive integers for \( \alpha, \beta, \ldots \) are included for the averaging procedure; the resulting FORTRAN program is not completely optimized. Besides improvements of this type, we shall consider an expansion of DEQTRAN to multidimensional ADI methods and possibly the implementation of solutions for ordinary differential equations. In addition, we shall link DEQTRAN to the numerical step, so that the numerical results are obtained directly when the necessary data are supplied.
General ideas of structured programming

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A feeling of insufficiency of present programming methods is now becoming more widely spread among all those involved in computing, especially in computer science. To the end of 1960s it was commonplace to speak about a "software crisis" those were beginnings of a critical appreciation of all features of programming in the effort to find one or more offenders of the difficulties: except goto statement there were complaints about floating-point calculations, global variables, semaphores, pointer variables, procedures with more than one output parameters and even assignment statements. It is clear that rejecting all these features, which would be no easy thing anyhow, would simplify programming to such an extent that there would be no trouble, but the solution of real problems would be either complicated or impossible at all.

I suppose that now - in the middle of the 1970s - having gone through the mess when almost everything was questioned and, at the same time, having faced up to the substantial changes in methods and organization of programming, is the best moment to present a survey of the conclusions of the contemplations and discussions going on in computer science for many years to the users of computers among physicists.

"Structured programming" received its name from the guidelines for using limited control structures: e.g. IF-THEN-ELSE and DO-WHILE. The following rules of structured programming could be considered the most important from the given point of view:

- Complete or partial elimination of the GOTO statement (its replacement by the constructs IF-THEN-ELSE and DO-WHILE).
- Program modularity.
- Top-down approach (stepwise refinement).
- Programming in chief programmer teams.

GOTO statement and other structures

It was proved that the following structures are sufficient for every program:

1. Simple sequence,
2. Selection,
3. Loop (cycle, repetition).

Their flowcharts are displayed in Fig. 1. They have a very important feature: for each of them there is only a single point of the transfer of control into the structure (entry) and similarly only a single point of the transfer of control out of the structure (exit). If the structures are combined in a program this feature (single entry - single exit) is retained, there being no back-tracking and thus the structures form natural blocks in the program which are the substance of good intelligibility and verifiability.

![Simple sequence](image1)
![Selection](image2)
![Loop](image3)

A program is called "structured" if only these three structures are used for all control functions. Most programming languages provide means for all three basic structures: e.g., PL/I has IF-THEN-ELSE and DO, FORTRAN has IF and DO, ALGOL has IF-THEN-ELSE and FOR etc. However, these languages moreover provide other structures "harmful" from the point of view of structured programming, e.g.

GOTO, RETURN. It means that structured programs may be written in any (perhaps in every) language but, on the contrary, every present programming language enables write unstructured programs as well.

Normally the programmer cannot confine himself to the observation of only the theoretically pure rules as he is obliged to take into consideration the length and efficiency of his program. Unfortunately, programming purely with the aid of the three basic structures is more or less efficient; the degree of efficiency is depending on the program and on the language. Generally, structured
programs in the ALGOL-like languages are more efficient than the same programs in the FORTRAN-like languages.

Although three structures of Fig. 1 are sufficient they are not the most efficient. Therefore more structures in programming languages are introduced. The flowcharts of two of them are displayed in Fig. 2:

- IF L THEN A is a simple abbreviation of IF-THEN-ELSE that is useful in many cases where the ELSE - clause is vacant,
- UNTIL L DO A is equivalent to (A; WHILE L DO A) where L is negation of L,
- FOR STEP UNTIL DO is a very useful form of loop that can be, of course, realized by WHILE-DO.

![Flowchart](image)

**IF L THEN A**  **UNTIL L DO A**  **FOR I:=J STEP K**  **UNTIL C DO A**

**Fig. 2**

There are cases when the elimination of GOTO - statement in the languages used at present means not only a loss of efficiency but even intelligibility, which is against the most important requirement of structured programming. The programmer using the present programming languages, e.g. FORTRAN, should avoid the occurrence of GOTO - statement as much as possible, i.e. should use it only on behalf of efficiency or better intelligibility.

**Program modularity**

The other features (rules) of structured programming are not so questionable as the problem of GOTO - statement and seem to be more natural and fairly simple. However, sometimes they are more important than strict elimination of jumps. This is true specially in the case of the rule of program modularity. The readability of the program is better and its changes are more feasible if the parts of a program are clearly differentiated, visibly separated and then grouped into modules according to the rule that the parts of a module are relative. Formally, it is advisable to limit the size of a part (subroutine) in a manageable bounds - approximately 50 lines.

**Top-down approach**

It means that design and implementation should proceed from the top to the bottom and it represents an efficient method of specification of program logic and structure.

**Chief programmer team**

This feature of structured programming has a significance only for programs of huge size - ten thousands of lines - that are written in a team and there it is very important and efficient.

**Other rules of structured programming** are according to my opinion less important for a physicist programmer, e.g. rule of simplicity, rule of good arrangement, verification of program, principles of abstraction, structuring of data. But structured programming is no miracle but it can contribute to some (or a great!) extent to the better programming. The understandability of a program depends mainly on the style of programming and the style cannot be expressed in any set of rules. If there is no style in a program (and in a programmer) the program will not be readable, good etc. even if it were a structured one.

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Abstract: A way of automatically converting a set of nonlinear, time-dependent, partial differential equations into a system of linear algebraic equations and an economical algorithm for solving such systems are described. The method is demonstrated for a nonlinear diffusion equation and the equations of hydrodynamics.

1) Introduction

The most commonly used methods in computing two-dimensional MHD flows are based on ADI solutions. For example, Lindemuth [1] has clearly described the necessary formalism. As is well known, ADI gets into difficulties when mixed derivatives appear. In special problems it is possible to diagonalize the transport tensors, as done by Anderson [2], who then applied a "fractional-step method", described in the book of Yanenko [3]. The aim of this paper is to explain the automatic generation of difference approximations with high spatial resolution and a complete implicit formulation in time, applicable to a wide class of problems, without the above restrictions.

2) The discretization-process

Differential equations are defined in infinitesimal intervals. The common difference approximations have similar properties. Their solutions converge to the exact one if \( \Delta t \) and \( \Delta x \) go to zero. For higher accuracy more grid points and smaller time steps become necessary, thus strongly increasing the computing costs. To overcome the first problem an integral formulation for each grid cell is used: The variables are represented by bilinear interpolation functions, and then the differential equations can be integrated over a grid cell, leading to nonlinear expressions in the grid values. Centering and linearization in time yields the coefficients for the variables at the new time. These two steps are performed by a computer-program in the symbolic language REDUCE, creating FORTRAN statements which can be used directly in the program for solving the algebraic equations.

3) Solution of the algebraic system

Implicit difference equations for MHD models are similar to those in elliptic problems. But the large sparse matrices have to be calculated and inverted for each time step. This might be done by direct Gauss-Seidel elimination - intolerable for large systems - or by SOR methods, probably changing the consistent implicit formulation.

To get a direct solution, the variables are arranged into row vectors for one space dimension, and then recursion formulas between block tridiagonal matrices have to be solved. This allows the use of external storage and, with special algorithms, keeps the "fill in" as small as possible, thus strongly reducing the number of operations, the main storage, and the computing time per time step.

4) Two test problems

Most MHD models contain hyperbolic and parabolic elements. Therefore, a diffusion equation and a hyperbolic system are treated with the formalism described above:

a) Nonlinear diffusion

Heat conduction in a fully ionized plasma without magnetic fields and sources is described by the equation

\[
\frac{\partial T}{\partial t} - \text{div} \left( \frac{T^2}{2 \eta} \frac{\partial T}{\partial x} \right) = 0.
\]

Fig. 1a) shows the initial conditions of a test problem on a 40 x 40 grid. Figs. 1b) and 1c) show the situation after 15 and 25 time steps respectively. The solutions of various ADI methods were compared with these results: Only a strictly "fractional-step method" leads to comparable values of the time step and energy conservation in this problem. But when mixed derivatives or other cross terms occur, implicit solution as described above should be preferable.

b) Compressible hydrodynamics

Next the equations of hydrodynamics

\[
\frac{\partial n}{\partial t} + \text{div} (nv) = 0, \quad \frac{\partial v}{\partial t} + \text{div} (nvv + \nabla (nT)) = 0,
\]

\[
\frac{\partial T}{\partial t} + \frac{1}{3} (nT) + \text{div} (nTy) + \frac{2}{3} nT \text{div} (v) = 0
\]

were tested. The system is solved on a 30 x 30 grid with reflecting walls, the initial density distribution being shown in Fig. 2a). After 15 time steps (Fig. 2b)) the spherical expansion has reached the walls, and after 30 steps the wave has been reflected by the walls and decayed into several waves, running into the center again. The limit for the time step of the implicit solution was compared with the von-Neumann stability condition, which in the problem considered could be exceeded by a factor of 2-4. Higher values did not produce numerical instability but decreasing accuracy most evident in the temperature evolution.

References:

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HYMNIA BLOCK, AN EFFICIENT EIGENVALUE SOLVER
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Abstract: Huge eigenvalue problems of the type \( \mathbf{A}\mathbf{x} = \lambda \mathbf{B}\mathbf{x} \), where \( \mathbf{B} \) is positive definite and \( \mathbf{A} \) is symmetric, are solved by an inverse vector iteration. Both matrices have a block structure.

1. Introduction

Two-dimensional stability problems such as those solved in [1,2] lead to eigenvalue problems of the type

\[
(\mathbf{A} - \lambda \mathbf{B})\mathbf{x} = 0,
\]

where \( \mathbf{B} \) is positive definite and \( \mathbf{A} \) can either be symmetric or hermitian. Both matrices are block structured and are stored on disk. HYMNIA BLOCK solves such an eigenvalue problem by an inverse vector iteration [3]. The method converges versus the eigenvalue with the lowest absolute value, but with an eigenvalue shift \( \lambda_0 \) it is possible to get anyone of them. In the calculations we make use of the block structures as well as of the symmetry of the matrices \( \mathbf{A} \) and \( \mathbf{B} \). In a given moment, only the upper part of one block of \( \mathbf{A} \) or \( \mathbf{B} \) is in the core memory. All operations are performed within this one dimensionally stored memory space. By decomposing \( \mathbf{A} \) before the iteration, only a number of point operations proportional to the number of matrix elements in the blocks is necessary for the iteration part. We also know which eigenvalue has been calculated. The eigenvalue is calculated in two different ways. The difference between the normalizing factor and the Rayleigh quotient gives an idea of the numerical error of the decomposition of \( \mathbf{A} \). This package can also be used simply to decompose \( \mathbf{A} \) or \( \mathbf{B} \) or to solve a system of linear equations.

2. Shift and decomposition of \( \mathbf{A} \)

To find all eigenvalues, a shift by \( \lambda_0 \) has to be performed. Instead of problem (1) we solve

\[
(\hat{\mathbf{A}} - \hat{\lambda}\mathbf{B})\mathbf{x} = 0,
\]

where \( \hat{\mathbf{A}} = \mathbf{A} - \lambda_0 \mathbf{B} \) and \( \hat{\lambda} = \lambda - \lambda_0 \).

Whenever all principle submatrices of \( \mathbf{A} \) are regular, we precede the iteration by a unique decomposition

\[
\hat{\mathbf{A}} = \mathbf{L}\mathbf{D}\mathbf{L}^H.
\]

Here, \( \mathbf{L} \) is a left hand side triangular block structured matrix containing unities on the diagonal. \( \mathbf{L}^H \) is its transposed and conjugate and \( \mathbf{D} \) is a diagonal matrix. It can be simply shown [4] that the number of negative terms of the diagonal matrix \( \mathbf{D} \) is equal to the number of negative eigenvalues of the problem (2). This information enables us to know which eigenvalue we have obtained. Numerically one block after the other is read from disk, decomposed, and restored on disk. All operations are performed within the upper half part of the block. Basically, we use an array of only \( m(m+1)/2 \) elements. The computing time for the decomposition is proportional to \( n^3 m^3/6 \). On a CDC 6500 one point operation takes about 20 \( \mu \)sec for real \( \hat{\mathbf{A}} \).
3. Inverse vector iteration

We start the iteration process
\[ LDL^H x_{k+1} = Bx_k \]  \hspace{1cm} (4)
by a randomly chosen initial vector \( x_0 \). One iteration loop consists in the following steps:

1. Multiply \( u_k = Bx_k \)  \hspace{1cm} (5)
2. Solve \( Lx_{k+1} = u_k \)  \hspace{1cm} (6)
3. Solve \( Dx_{k+1} = y_{k+1} \)  \hspace{1cm} (7)
4. Solve \( L^H x_{k+1} = y_{k+1} \) \hspace{1cm} (8)
5. Normalize \( x_{k+1} \). \hspace{1cm} (9)

The resolutions 2, 3, and 4 are simple back substitutions. The number of point operations is proportional to \( n^2 \). For a CDC 6500 one point operation for the iteration takes about 100 us. The iteration process is stopped when each vector component has converged in between a given range. For an adequate numerical treatment, we store the blocks on one disk area such that only sequential reading has to be performed in one iteration step.

4. Eigenvalue

The eigenvalue is first obtained from the normalizing operation of Eq. (9). Starting again from the original matrices \( A \) and \( B \) the Rayleigh quotient
\[ \lambda = \frac{(x^H, Ax)}{(x^H, Bx)} \]  \hspace{1cm} (10)
gives a second eigenvalue. For a converged process (Eqs. 5 - 9) the difference between these two eigenvalues gives an information about the accuracy of the decomposition in Eq. (3). For a non-converged iteration the two values will differ and they can be used for a new guess of \( \lambda_0 \).

5. Examples

This block structured eigenvalue package has been constructed for very big eigenvalue problems. In a specific problem, the eigenvector \( x \) consists in 2520 components (\( n = 20 \) and \( m = 168 \)). For 10 iterations the HYMNIA BLOCK package takes 10' on a CDC 6500 to find one eigenvalue and its corresponding eigenvector. For this case, the size of one matrix block was \( m(m + 1)/2 = 14196 \) and the total number of elements in the 20 blocks was equal to 266763.

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References

Application of Nonlinear Optimization to
Solution of Differential and Integral Equations

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With progress of large-scale high-speed computers many kinds of nonlinear optimization algorithms have been developed and used in the fields of engineering and sciences. In brief the nonlinear optimization is a procedure to minimize (or maximize) an appropriately chosen nonlinear objective function of a set of unknown parameters \( \phi(x) \). By choosing a most appropriate objective function for the purpose one can determine values of unknown parameters effectively. The procedures are applicable to a wide variety of problems. Applications to the curve fitting and process control of such as a chemical plant are well investigated examples.

The usual curve fitting problem is to determine a set of unknown parameters from experimental data by assuming a functional dependence of the parameters explicitly. This kind of procedures is easily generalized to those where a functional dependence of the unknown parameters is not given explicitly. Solutions of differential or integral equations are the typical examples of this kind. In the following we describe the application of the nonlinear optimization to the solution of differential or integral equations which appear in the field of nuclear fusion research.

An integral equation of the first kind often appears in the analysis of experimental data such as in the plasma density measurement by microwave phase shifts or in the spatially resolved spectroscopic measurements of a plasma column. When the equation is of the Fredholm type, it is given as

\[
\phi(x) = \int_a^b dy f(y) K(x,y),
\]

where \( \phi(x) \) and \( f(y) \) are the experimental data and the function to be determined, respectively, and \( K(x,y) \) is the kernel of the equation. As well known the integral equation of the first kind is an ill conditioned problem and one can obtain a solution only by appropriate choices of a mathematical model of the problem. Therefore, to solve the equation is reduced to a problem how to choose the most appropriate mathematical model to each equation. This problem is solved easily by optimizing the following objective function by using a computer code of the nonlinear optimization,

\[
F = \int_A dx \left( \phi(x) - \int_a^b dy f(y) K(x,y) \right)^2 \omega(x).
\]

Once the equation is reduced to the optimization problem where the form of the function \( f(y) \) is represented by an arbitrary set of unknown parameters, minimization of the objective function \( F \) with respect to these unknown parameters gives the solution of the integral equation. One of the advantages of the method lies in the fact that constraining conditions on the solution are easily incorporated. By an appropriate choice of the conditions and a set of unknown parameters one can suppress unphysical oscillations which are apt to appear in solutions of the integral equation of the first kind. This procedure is not restricted to the equations of the Fredholm type or even to linear integral equations.

Another application is the solution of differential equations. This procedure is especially effective when the equation is solved by the finite element method. Usually in the finite element method a functional whose Euler equation is the original differential equation is solved by using a set of local base functions and then the equation is solved by reducing the functional to simultaneous linear equations. By using the nonlinear opti-
mization, however, the solution of the equation is directly obtained by mini-
mizing the functional with respect to the parameters which characterize the
solution[3]. In the former method some appropriate iteration procedures are
required when one is to solve a nonlinear differential equation, but in the
latter, the iteration procedure necessary to solve the nonlinear differential
equation is already included in the nonlinear optimization algorithm which is
considered as a black box for the user. In the following we present the
solution of the equation for the magnetohydrodynamic equilibrium of a toroidal
plasma as an example of solution of partial differential equations by using the
nonlinear optimization. The well known differential equation for the problem
is given in a cylindrical coordinate \((r, \psi, z)\) as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = -\frac{1}{\mu} \frac{d I_A}{d \psi}, \quad \text{in plasma,}
\]

\[
= 0, \quad \text{in vacuum,}
\]

where \(\psi\) is the magnetic flux function, \(p = p(\psi)\), the pressure of the plasma
and \(I_A/r\) is the toroidal magnetic field (\(B_\psi\)). The functional for the Euler
equation (Eq.(3)) is easily derived[4] as

\[
L = 2\pi \int_0^L \left\{ \frac{1}{2} \left[ \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \psi}{\partial z} \right)^2 - \left( \frac{I_A}{r} \right)^2 \right] - p \right\} r \, dr \, dz.\quad (4)
\]

The solution can be obtained by dividing the poloidal plane into a number of
finite elements, carrying out the above integration and minimizing the
functional \((L)\). Generally this is a free boundary problem and \(\psi\) is constant
on the boundary between the plasma and vacuum. In such a case it is convenient
to adopt the positions of the contours of \(\psi\) as independent variables instead
of the values of \(\psi\) for the prescribed positions. As the functional (Eq.(4))
is a very complicated nonlinear function of the positions, use of the nonlinear

optimization is very effective to solve this kind of problems. Figures 1 and
2 show the examples of the results for a fixed and free boundary problems,
respectively. These results were obtained by using the simplex type nonlinear
optimization algorithm by Nelder and Mead[5].

In summary the nonlinear optimization can be applied successfully to the
solutions of differential and integral equations. It is especially useful
for a relatively small nonlinear problem because it makes the numerical formul-
ation very simple. In future it will become also useful for a large-scale
problem as the speed of computers is expected to become faster and algorithms
of the nonlinear optimization will be refined.

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[1] For example, J.Kowalik and M.R. Osbourne, Methods for Unconstrained Opti-

Fig.1: Initial (broken lines) and
final (solid lines) positions of the
elements for a fixed boundary
problem \((B=0.9 \, m, r_p=0.30 \, m, p(\psi)=0)\).

Fig.2: Contours of \(\psi\) for a free boundary
problem \((B=0.9 \, m, r_p=0.25 \, m, p(\psi)=0)\).
Plasma vacuum boundary is denoted by
\(\psi=0.0\).
Inverse iteration in MHD equilibrium computations.

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K. Lackner and K.U. v. Hagenow developed a fast iteration scheme for computing axisymmetric MHD equilibrium configurations \[ \text{[6]}, \text{[3]}, \text{[4]} \]. A mathematically more detailed analysis of its convergence properties is presented here.

If axisymmetry is assumed and the ansatz \( r \vec{B} = f \vec{e}_\theta \times \vec{e}_\phi \) is used for the magnetic field \( \vec{B} \), ideal MHD equilibria are described by solutions of the equation

\[
(1a) \quad - \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{1}{r^2} - \frac{2}{3} \frac{\partial^2 \psi}{\partial z^2} + 4 \pi r^2 \frac{dp}{d\psi} + F \frac{df}{d\psi} = f(r, z, \psi)
\]

for the flux function \( \psi \). The plasma pressure \( p \) and \( F \) can be shown to be functions of \( \psi \) only. The differential operator \( -\frac{\partial^2}{\partial r^2} \) is uniformly elliptic since \( r \geq r_0 > 0 \) in the torus. Equation (1a) has to be supplemented by some boundary conditions, in the simplest case by the Dirichlet condition

\[
(1b) \quad \psi/\psi_D = 0.
\]

It is quite possible for eq. (1) to have several solutions. This is the case if, for example, it is assumed that \( f(r, z, \psi) = -\psi \phi \) \[7\].

It has been shown by several authors (see \[3\] and \[8\]) for references) that certain solutions cannot be computed with the common iteration scheme

\[
(2) \quad \psi^{n+1} = f(r, z, \psi^n) \quad \psi^{n+1}/\psi_D = 0.
\]

Another iteration scheme

\[
(3) \quad \psi^{n+1} = f(r, z, \psi^n) \quad \psi^{n+1}/\psi_D = 0.
\]

was therefore developed \[6\], \[3\], \[4\]), which has been highly successful. In this scheme the functional form of the r.h.s. of eq. (1) is also modified from iteration to iteration. These modifications of \( f \) are such that physical quantities depending on \( \psi \) assume a prescribed value \[6\], \[3\].

In the ideal case just as many side conditions are imposed as will make the problem uniquely solvable. In order, however, to prevent the problem from becoming unsolvable as a result of overdetermination, it is advisable to compensate by introducing parameters, which also have to be computed by the iterative process \[4\].

The general procedure is illustrated \[4\] by the well known example

\[
(4) \quad \psi^{\infty} = \lambda r^2 \exp \psi \text{ in } D, \quad \psi = 0 \text{ on } \partial D.
\]

Here \( \max |\psi(r, z)| = M \) is prescribed, and the iteration is obtained by

\[
(5) \quad \psi^{n+1} = \lambda \psi^n \exp \psi^n \text{ in } D, \quad \psi^{n+1} = 0 \text{ on } \partial D,
\]

\[
(6) \quad \psi^{n+1} = \lambda \psi^n + M, \quad ||\psi^n|| = M, \quad \psi^{n+1} = 0 \text{ on } \partial D.
\]

If one replaces eq. (5) by

\[
(6a) \quad \psi^{n+1} = \psi^n + M, \quad ||\psi^n|| = M
\]

\[
(6b) \quad \psi^{n+1} = \lambda r^2 \exp \psi^n \text{ in } D, \quad \psi^{n+1} = 0 \text{ on } \partial D,
\]

\[
(6c) \quad \psi^{n+1} = \frac{M}{||\psi^n||} \lambda \psi^n, \quad ||\psi^n|| = M
\]

\[
(6d) \quad \psi^{n+1} = \lambda \psi^n \exp \psi^n, \quad ||\psi^n|| = M
\]

the new \( \psi^{n+1} \) differ from the old ones by a scalar factor, but the \( \psi^n \) and \( \lambda \) remain unchanged. From now on we shall therefore deal with the simplified iteration scheme (6), which is a straightforward generalization of a common method for linear eigenvalue problems called "inverse iteration". Its convergence to the smallest
eigenvalue and the corresponding eigenfunction is discussed in \[ \text{[9]} \]
§21 (matrices and vectors), \[ \text{[2]} \] §14 (partial differential equations
\[ \text{[1]} \], §2 \( - h(x) \varphi = \lambda h(x) \varphi \)) the continuous function \( h \) is now
allowed to change sign in \( D \), as may occur for a sufficiently
diagonally plasma in a torus where the inverse aspect ratio \( a/R \)
is relatively large.

In the linear case, the only possible of the normalization (6d) is
for an overflow \( \lim_{n \to \infty} \left| \lambda_n \right| > 1 \) or an underflow \( \lim_{n \to \infty} \left| \lambda_n \right| < 1 \).

As was mentioned earlier and may be seen in the proof of the follow-
ing theorem, normalization is an essential tool for obtaining
convergence to nontrivial solutions in the nonlinear case.

**Theorem:** Let \( D \subset \mathbb{R}^2 \) be a bounded, connected, open set with smooth
boundary \( (\mathbb{R} \times C^2) \), \( 0 < \alpha < 1 \). Let \( L \) be uniformly elliptic
with \( H^{1,\text{a}} \) continuous coefficients \( A_{ij}, b_i \)
(7) \( L u = A_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b_i \frac{\partial u}{\partial x_i}, \quad (A_{ij}) \text{positive definite in } D. \)

Let \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfy
(\( F.1 \)) \[ |f(x_1, u) - f(x_2, u)| \leq c_1 |x_1 - x_2|^{\alpha} \]
(\( F.2 \)) \[ |f(x_1, u) - f(x_2, u)| \leq c_2 |u_1 - u_2|, \]
(\( F.3 \)) \[ 0 \leq f(x, u) \leq M \]
for \( x, x_1, x_2 \in \mathbb{R}, 0 \leq u, u_1, u_2 \leq M > 0 \) fixed.

Then all iteratives generated by the scheme
(\( 8a \)) \[ u_0 \in \mathbb{R}^{\mathbb{R}}, \quad \|u_0\| = M, \]
(\( 8b \)) \[ u_1 = f(x, u_0) \text{ in } D, \quad u_{n+1} = 0 \text{ on } \partial D, \]
(\( 8c \)) \[ \lambda_{n+1} = \frac{M}{\|u_{n+1}\|}, \]
(\( 8d \)) \[ u_{n+1} = \lambda_{n+1} u_{n+1}^n \]
exist. The sequences \( \{u_{n+1}\}_{n \in \mathbb{N}} \) and \( \{u_n\}_{n \in \mathbb{N}} \) have at least one accumulation
point \( \hat{u} \) and \( u \) respectively in \( C^2(\mathbb{R}) \). If \( u \) is the limit function of
a subsequence \( \{u_{n_k}\}_{k \in \mathbb{N}} \), it is a solution of
(\( 9 \)) \[ L u = \lambda f(x, u) \text{ in } D, \quad u = 0 \text{ on } \partial D, \]
with \( \lambda = \lim_{k \to \infty} \lambda_{n_k} \).

**Corollary:** Let \( z \) be the solution of
(10) \[ L z = 1 \text{ in } D, \quad z = 0 \text{ on } \partial D. \]
Then a priori bounds for \( \lambda \) are given by
(11) \[ \frac{M}{k_2 \|z\|} = \lambda \leq \frac{M}{k_1 \|z\|}, \]

**Remark:** If it is known that eq. (9) has a unique solution \( u \) satisf-
ifying \( \|u\| = M \), the theorem ensures convergence of the total sequence.

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NUMERICAL AND ANALYTICAL ALTERNATIVE TO THE KRYLOV BOGOILUBOV METHOD
APPLICATION TO SLIGHTLY NONLINEAR AUTONOMOUS AND NONAUTONOMOUS SYSTEMS

INTRODUCTION

We are dealing with slightly nonlinear harmonic oscillators described by \( x = x_0 + c x_1 + c^2 x_2 + \ldots \). The philosophy of the method has been indicated in [1] [2] and [3]. Firstly this method may be considered as a numerical scheme with a "giant" time step \( T \) after which we reinitialize the different terms of the series (here a Poisson series) as in all numerical algorithms. In that case \( T \) can be adapted to the smallness of \( \epsilon \) (which characterizes the perturbing linear and nonlinear terms). The important point is that the method is still working for finite \( \epsilon \) although in that case the step \( T \) must be taken small.

In a more analytical version the method is combined with a stroboscopic scheme. The time step \( T \) is taken nearly equal to \( 2\pi \) and successive values of the maximum are computed (and of course of the times at which these maximum occurs). The computation of these maxima can be viewed as a discretized version of the slowly varying amplitude in the Krylov Bogoliubov method. An obvious advantage is that our method gives a physically simple operational definition for the "quasi period" (it is the interval between the \( n^{th} \) and the \( (n+1)^{th} \) maximum given as a function of the amplitude of the \( n^{th} \) maximum).

NONAUTONOMOUS SYSTEM

Let us consider Equation [1] with

[2] \( f(x, \dot{x}, t) = E \cos (ut + \phi) + k x^3 \)

We notice that the presence of \( \epsilon \) in front of \( f(x, \dot{x}, t) \) implies that both the forcing term and the nonlinear term are small which means that the system is always nearly oscillating at its own frequency. We briefly recall the steps of the solution.

We introduce the series

\[ \begin{align*}
  x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \\
  \dot{x} = \dot{x}_0 + \epsilon \dot{x}_1 + \epsilon^2 \dot{x}_2 + \ldots
\end{align*} \]

and identifying terms in \( \epsilon^0 \) and \( \epsilon^2 \) we obtain the hierarchy equation

[3] \( \begin{align*}
  \ddot{x}_0 + x_0 &= 0 \\
  \ddot{x}_1 + x_1 &= E \cos (ut + \phi) + k x_0^3 \\
  \ddot{x}_2 + x_2 &= 3 \dot{x}_0 \dot{x}_1 + \epsilon x_2^2
\end{align*} \)

Equations 3 are successively solved, the initial conditions being completely absorbed by \( x_0 \) and \( \dot{x}_0 \), more precisely

\[ \begin{align*}
  x_0(0) &= x(0) \\
  \dot{x}_0(0) &= \dot{x}(0) \\
  x_n(0) &= \dot{x}_n(0) = 0 \quad \text{for} \quad n > 1
\end{align*} \]

If we suppose that at time \( t = 0 \) \( \dot{x}(0) = 0 \) and \( x(0) = \Delta \) since \( \epsilon \) is small we know that the next maximum will take place around \( t = 2\pi \). More precisely for \( t = T \) with

\[ T = 2\pi \left(1 + \epsilon \eta_1 + \epsilon^2 \eta_2 + \ldots\right) \]

Introducing \( T = 2\pi \left(1 + \epsilon \eta_1\right) \) in \( \ddot{x}_0 + x_0 = \epsilon \dot{x}_1 + \epsilon^2 \dot{x}_2 = 0 \) we obtain \( \eta_1 \), then \( \ddot{x}_0 + \epsilon \dot{x}_1 + \epsilon^2 \dot{x}_2 = 0 \) with \( T = 2\pi \left(1 + \epsilon \eta_1 + \epsilon^2 \eta_2\right) \) gives \( \eta_2 \) etc...

Introducing \( \eta_1 \) into the solution \( x(T) \) we get the new maximum we notice that what is really important is the new phase of the forcing term \( ut + \phi \).

MAIN RESONANCE

We suppose \( \omega = 1 \) and obtain the first order set of Equations for the amplitude \( A_n \) and the phase \( \theta_n \) with \( f(x, \dot{x}, t) = E \cos (ut + \phi) + k x^3 \)

\[ \begin{align*}
  \frac{\dot{A}_n}{A_n} + 1 &= A_n + \epsilon E \sin \theta_n \\
  \frac{\dot{\theta}_n}{2\pi} + 1 &= 2\pi E \left(\frac{E}{2A_n} \cos \theta_n + \frac{3kA_n^2}{2A_n} \right)
\end{align*} \]

The results of the K.B. method [4] page 241, can be recovered easily as the adiabatic limit of [5]. If we let \( \epsilon \rightarrow 0 \) and consider a new time variable such that \( t = \epsilon t \) we can formally write \( \dot{t} = 2\pi \epsilon \), \( A_n + 1 - A_n = \Delta A \) and \( \dot{\theta}_n + 1 - \dot{\theta}_n = \Delta \phi \).

Then:

\[ \begin{align*}
  \frac{\Delta A}{\Delta t} &= \frac{E}{2\pi} \sin \phi \\
  \frac{\Delta \phi}{\Delta t} &= \frac{E}{2A_n} \cos \theta_n + \frac{3kA_n^2}{2A_n}
\end{align*} \]

Of course it would be a mistake to think that [6] is an improvement on [5] and that by solving precisely [6] (for example with a very precise numerical method and a very small time step \( \Delta t \)) we gain accuracy by comparison with [5].

The second order (in \( \epsilon \)) approximation is: \( \Delta = 2\pi \epsilon \)

\[ \begin{align*}
  \ddot{A}_n + 1 = A_n + \Delta \left( \frac{E}{2A_n} \cos \theta_n + \frac{3kA_n^2}{2A_n} \cos^2 \theta_n + 4\pi \frac{\dot{\theta}_n}{n} \right) \cos \theta_n + \left(\frac{3kA_n^2}{6A_n} \right) \sin \theta_n - \frac{A_n^2}{6A_n} \sin^3 \theta_n \cos \theta_n
\end{align*} \]

\[ \begin{align*}
  \dot{A}_n + 1 &= A_n + \Delta \left( \frac{E}{2A_n} \sin \theta_n + \epsilon \left(\frac{E}{2A_n} \cos \theta_n + \frac{3kA_n^2}{2A_n} \right) \right)
\end{align*} \]
Fig. 1 shows the evolution of the oscillator in the phase space plane of cylindrical coordinate $A \theta$. The oscillator is synchronised with the forcing term as evidenced by the fact that the representing point does not circle the origin and evolves on a closed curve. The very long cycle associated to this curve ($2020 \times 2\pi$) indicates a phase modulation of amplitude $42^\circ$. Notice that we followed the evolution of the system during 3000 periods. The accuracy of the results were obtained by running the numerical version of "the giant step method" reinitialising the equation 10 times per period.

**Influence of a damping term around the main resonance**

Now we take $f(x, \dot{x}, t) = \xi \cos(\omega t + \phi) - \beta \dot{x} + \kappa x^3$. We are interested to know if the system is going to "lock" to the external frequency $\omega$.

Obviously $\omega$ must be close to one. To first order in $\epsilon$

$$\dot{\phi}_n + \omega = \phi_n + 2\pi \omega + 2\pi \epsilon \omega \left(\frac{3}{8} \kappa A_n^2 \right) + E \frac{\omega}{2 \pi A_n^2} \left(\sin \phi_n \sin (2\pi \omega + \phi_n)\right)$$

$$\dot{A}_n + \epsilon \left(\frac{E}{1 - \omega^2} \left(\cos (2\pi \omega + \phi_n) - \cos \phi_n\right) = \beta \pi A_n\right)$$

On fig. 2 we show how a damping term brings a complete "locking" for $\omega = 1$ (the phase modulation disappearing). The transient behaviour of the system is followed during 3000 periods.

Fig. 3 exhibits the behaviour of the system when $\omega = 0.99915$. We see two possible cycles. On the first one (the large closed curve on Fig. 3) starting as in Fig. 1 we nearly close the loop around the origin. But finally the oscillator "locks" on the driving frequency $\omega$ with a rather large phase and amplitude modulation. The period of this cycle is $4.270 \times 2\pi$. Another cycle also indicated on fig. 3 is possible starting with $\phi = 0$ and the same amplitude. Notice that both cycles are computed for $\beta = 0$.

The interesting point is that a damping term connect the two cycles. Fig. 4 shows what happens starting from the same initial conditions which in Fig. 3 give the large cycle. The behaviour is practically the same for the first 200 periods. Then the damping brings the trajectory upon the second (small) cycle with a "locking" not quite perfectly completed after 4500 periods.

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potential arising from the given interior charges, together with a set of boundary screening charges. The latter may be calculated from the potential. We find the corrections needed to restore the given boundary charges and perform a direct convolution for the boundary potential. Thence we calculate the supplements required in the interior as for the case of a prescribed boundary potential. We perform 6 SFT's in all. For a problem on a $33 \times 33 \times 33$ mesh these require nearly half the total calculation time and the speed is about three times that of the direct method. For larger problems, the ancillary calculations are less important and the speed gain is greater. The store requirement is that for the original mesh plus a much smaller amount of working space, or about half that needed for the direct method.

The simplest finite difference representation for the Laplace operator relates a charge at a mesh point to the potentials at that and at neighbouring mesh points. A fourth difference representation involves more distant points and would lead to a double layer of screening charges at and outside the boundary. However, we may rearrange the operator to avoid this, without losing fourth difference accuracy. For a mesh with equal intervals in each direction, we can extend the procedure to give sixth difference accuracy. Each operator corresponds to a distinct gravitational Greens function, with leading error term $\mathcal{O}(R^{-n-1})$, where $R$ is the geometrical distance from the source, and $n$ is the highest order difference retained in the Laplace operator. On a $33 \times 33 \times 33$ mesh the solution time on the
CDC 7600 is between 2 and 2.5 secs, depending on the representation chosen for the Laplace operator.

The procedure is equivalent to convolution with the Greens function appropriate to the Laplace operator used, and we are not free to choose the Greens function arbitrarily as in the direct convolution process. This has led to our considering the question of mass assignment schemes to mimic continuous source distributions as accurately as possible. These distributions occur for example in hydrodynamical problems. We are also exploring the possibility of treating the mass distribution of a galaxy as a continuous distribution. We follow the motion of a representative sample of normal mass stars and use this sample to determine the low order moments of the galactic mass distribution around each mesh point. These moments define a continuous cloud within a cell centred on a mesh point.

The potential field due to such a cloud may easily be calculated. We have found schemes for assigning masses to 27 mesh points in 3-D in such a way as to mimic the potential arising from uniform clouds (matching the monopole moment) and clouds with a linear density gradient (matching the dipole moments). Kuer, Dawson and Rosen (1972) have used a similar technique in particle simulations, but they match the dipole moments of the mass distribution rather than the actual potentials. The potential field generated by the Poisson solver at any grid point is an extremely good approximation to the true Newtonian field of the clouds at a distance, but less so far the local contribution. However, the error in the local contribution is at worst 0.5%. This could be reduced by employing more masses for the representation of each cloud, and the clouds could be further smoothed by including higher order moments (thus reducing grid noise), but it is doubtful that the improvement will be sufficient to justify the increased computing costs.

For discrete source distributions, for example small star clusters, the field of an individual point mass may be reproduced by introducing a set of masses over the near grid points. These are chosen so as to eliminate the lower order moments of the error introduced by a simple nearest grid point scheme.

References


Calculation of Neutral Beam Heating of Tokamaks

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High-energy neutral beams are planned for a number of large tokamaks, such as the TFTR, PLT, and PDX experiments at Princeton. We have modeled the effects of such heating using a one-dimensional tokamak plasma transport code. A series of calculations have then been done to explore various proposed injection methods. The possible effects of an impurity driven beam deposition instability were investigated.

The one-dimensional tokamak transport code models the plasma as a cylinder of infinite extent and treats the plasma parameters as a function of their minor radial coordinate. The electron transport is computed using an appropriate mix of micro-instabilities models. The ion and impurity transport is neo-classical. The effects of neutral gas cooling and charge exchange are included. The effects of impurities on the plasma transport and impurity radiation are also included. High-Z impurities sputtered from the wall by fast charge-exchange neutrals are part of the impurity transport model. The results of the code are in reasonable agreement with most current experiments.

Following the methods of J. Rome et al., the beam deposition profile is calculated at ten times during the injection pulse for each injector. The beam ions are then assumed to stick to the flux surface on which they were born. The beam ions are slowed down classically, heating the plasma electrons and ions, and, in the case of a deuterium beam, and a tritium plasma, react with the background plasma to produce 4 neutron and 3.5 meV alpha particles. Loss of the beam ions due to charge exchange with the background neutrals is included.

The calculations for PLT show that the experiment might be expected to reach a temperature in the range of 3.5 - 4 keV with 2.5 MW of injection. The results for PDX are similar.

The results for TFTR indicate that there may be a beam penetration problem for long injection pulses due to high-Z impurity build-up at the plasma edge. The neutral beams currently planned for TFTR are designed to deliver 20 MW of 120 keV deuterons, and 15 MW of 60 keV and 40 keV deuterons. At 120 keV, the beam is ionized primarily by ion impact ionization. Our best guess of the cross section for this process is $\sigma_p Z^2$ where $\sigma_p$ is the cross section for ionization of atomic hydrogen by protons, and $Z^2$ is the ionic charge of the target ion. Thus, small amounts high-Z impurities can greatly reduce the penetration of the neutral beam into the plasma. This reduced penetration will enhance the heating of the plasma edge, which will increase the flux of fast change exchange neutrals to the vacuum chamber wall, thereby increasing the sputtering of high Z-wall material. This raises the average Z at the edge of the plasma, and reduces the beam penetration even further. This "instability" can thus limit the useful pulse length of a neutral beam injection.

On TFTR, we found that this effect may limit the useful injection pulse length to the order of 100 ms. If the 60 keV and 40 keV components of the beam are eliminated, the useful injection pulse length is extended to 200 ms or more.

Further calculations of neutral beam heating are currently being carried out using a Monte Carlo code which integrates the guiding center orbit equations for fast ions in a tokamak. This ions are slowed down and pitch angle scattered classically. This technique will enable us to investigate orbit effects on the beam ions, such as trapping in the field ripple, and loss comes from large banana orbits. These questions are important for the nearly perpendicular injection necessary to penetrate tokamak plasmas in the mid-$10^{14}$ particles cm$^{-3}$ range.


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Effects of Impurity Transport on Neutral Injection in Tokamakks

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ABSTRACT

A 1-D radial transport code is used to examine the possible effects of impurity accumulation on high energy neutral injection heating in a Tokamak.

INTRODUCTION

Energetic neutrals produced in a plasma by charge exchange reactions can carry energy directly to the walls and sputter impurity atoms back into the discharge. Since sputtering yields generally increase rapidly in the 0 - 10 keV range there is some concern over the role of impurities in large future machines designed to operate at ion temperatures of several keV.

To achieve temperatures of thermonuclear interest (< 10 keV) Tokmaks will require supplementary heating; future machines will therefore be driven by large amounts of injected power, possibly through high energy neutral beams. In this case, the presence of impurities in the plasma will influence the power deposition profile by inhibiting the penetration of the beam to the central core. In this paper we examine some possible implications of impurity accumulation on neutral injection heating in a machine having the dimensions of the proposed JET experiment viz. a = 128 cm, R = 293 cm I = 3MA, B = 30 kG.

TRANSPORT MODEL

To calculate the plasma and impurity transport we use DUCHS 1-D radial transport code [1]. The ion and impurity transport are assumed to be neoclassical while the electron transport is pseudoclassical i.e. D_e = K_e = C \gamma(\zeta_{\text{eff}}^2) \nu_e \nu_n^2, where C is a constant \gamma \approx 10 and \gamma(\zeta_{\text{eff}}^2) = 0.29 + 0.457/(1.07 + \zeta_{\text{eff}}^2). Ionization parameters for the impurity ions, assumed to be \nu_e, are obtained using a model described by Noshel [2] which assumes that the impurity is everywhere in a state of ionization equilibrium. This assumption is valid if the timescale \tau over which the macroscopic plasma parameters change satisfies the condition \tau > 10^{12}/n_e (cm^-3) sec. For large future machines where diffusion timescales \approx 1 sec are expected this steady state model is appropriate.

BOUNDARY CONDITIONS

The DUCHS code includes an influx of cold neutral gas at the boundary together with some number of generations of hot neutrals produced by successive charge exchange collisions. However, for a machine of JET dimensions it is necessary to follow a large number of generations (> 10) to find neutrals which have sampled the entire temperature distribution. Since this consumed excessive computer time we have used instead a simple analytic description suggested by Engelmann [3]. In either case we can calculate the flux of escaping neutrals and their temperature distribution. Using experimental data on sputtering yields we can then determine the subsequent influx of heavy atoms released from the walls.

NEUTRAL INJECTION MODEL

We consider a neutral beam of energy \nu_o and current I_o incident on a cylindrical plasma at some angle \alpha to the magnetic axis. The number of beam particles deposited/sec in the radial interval dr is N_o dr where [4]

\[ N_o(r,t) = \frac{I_o}{e} \frac{dN_o(r)}{L(r)} \left[ \exp \left( -\int_a^r \frac{N_o}{L} \, dr' \right) - \frac{N_o}{L} \right] \]

and L(r) = 5.5 \times 10^{14} E_o \sin \alpha. Particles trapped at radius r are distributed over a magnetic surface giving a particle source term S = N_o/4\pi^2r^2; the singularity as r = 0 is avoided by giving the neutral beam a finite radius r_b so that S = N_o/4\pi^2r_b^2 if r \geq r_b.

At high energy (\nu_o > 30 keV) the cross-sections for charge exchange and impact ionization of the beam are roughly proportional to the effective Z for momentum transfer, Z_{\text{eff}} [5]. Thus, in our model, we include the effect of impurities by reducing L(r) by a factor Z_{\text{eff}} if \nu_o > 30 keV.

RESULTS

Figures 1 - 4 summarise the results of two calculations. In each case, the initial conditions are the same. That is, the temperature distributions are parabolic with peak values of 100 ev while the initial density profile is flat \alpha(1 - r/a)^2; the full current of 3MA is initially distributed parabolically throughout the plasma volume. In one case a hydrogen plasma is allowed to evolve by ohmic heating alone (dotted curves on figures) while in the second case a 4MW beam of neutrals at 70 keV is injected at 45° to the magnetic axis, after 0.5 sec. Figure 1 shows how the total impurity
concentration increases much more rapidly when the beam is switched on; this is simply a consequence of the increased sputtering due to higher transient temperatures in this case. Figure 3 shows the beam power deposition profile at two times. Figure 3(a) shows the deposition profile immediately after the beam is switched on. Although the profile is heavily attenuated by the impurities there is still significant penetration to the centre. However, as the impurity accumulation continues there is less penetration to the centre and more beam power is deposited near the edge (Figure 3(b)). Figure 4 which shows the time variation of the mean temperatures, illustrates the consequence of this effect. When the beam is switched on there is an initial increase in the mean temperatures. As the beam deposition profile narrows most of the injected energy is deposited in a region where the impurity density, N_e, is large (Fig.2). Eventually N_e becomes sufficiently large to dissipate all the beam power by impurity radiation and increased heat conduction. At this point the temperature drops rapidly to a value smaller than would have been obtained by ohmic heating alone. Some further implications of this result will be discussed.

REFERENCES


TURBULENT TRANSPORT IN REVERSED FIELD PINCHES

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1. INTRODUCTION

MHD stability of the Reversed Field Pinch (RFP) relies on reversal of the toroidal field component in the outer plasma region. Interest in this configuration comes from its potential economic advantages as a thermonuclear reactor, since compared to a Tokamak the RFP supports a higher value of B, the ratio between plasma and total magnetic pressure. We report results of computations on the time-evolution of the RFP using a 1D MHD model.

As initial conditions we choose a stable low-B RFP configuration forced during the setting-up phase of the pinch either by programming the wall fields [1] or as the end result of plasma turbulence [2]. The plasma is then ohmically heated by a sustained electric current so that its temperature and pressure increase. During this acceleration phase the RFP is expected to violate the Saydam-MHD stability criteria [3] on the maximum local value of pressure gradient - ∂p/∂r, especially near the centre of the plasma column where the magnetic shear is weak. In order to take the calculation further we need some assumptions about the effects of the instability. We suppose that within any Saydam-unstable zone there is a local 'turbulent' transport of heat and plasma across the magnetic field. This causes the pressure and temperature profiles to flatten and the departure from stability to be automatically reduced. Within a Saydam-stable zone we postulate that the confinement properties remain intact although the transport across such a zone will be enhanced by a steepening of the gradient to balance the flattening elsewhere. In some cases the pressure steepening may be enough to violate the stability criterion so causing the inter-zone boundary to move. The division into 'turbulent' and 'stationary' zones is analogous to the situation in stars [4] but requires justification in the MHD case (to be briefly discussed in §3).

A model of this kind was proposed by Saydam [3] in his 1956 Geneva Conference paper:

"The result seems to go qualitatively as follows: the mixing of a small unstable region leads to a distribution which is less unstable on the inside and more unstable on the outside. Thus, if any interior shell were unstable, it would mix until stable, and this in turn would stop. In this fashion such an instability would not set its own outward toward the surface." If, however, a layer near the surface is given excess stability, the outward progression of the mixing should be stopped. This excess stability of the surface layers ought to be insured if the B field were so programmed that u' is made quite large in the region. The simplest programming appears to be one which would reverse B in the vacuum after the plasma has pinched."

We investigate its quantitative behaviour for parameters appropriate to a 'next-generation' RFP device.

2. PHYSICAL AND NUMERICAL MODEL

A 1D cylindrical approximation is used with coordinates (r,θ,ψ). All functions depend only on r. The plasma, assumed to be hydrogen, is described by a particle density n and two temperatures T_e, T_i; the magnetic field components are B_r, B_θ, B_ψ. Diffusion processes dominate so that the plasma evolves slowly through a sequence of states in pressure equilibrium and the equation of motion reduces to

\[ \frac{Dp}{Dt} = \frac{1}{\rho} \frac{D}{Dr} \left( \rho \frac{Dv}{Dr} - p \right) \]

with p = \rho u^2 + \frac{1}{2} \frac{B^2}{\mu_0}. Equation (1) is solved at each timestep by adjusting the coordinates of a Lagrangian mesh [5]. Letting \frac{D}{Dt} denote a Lagrangian derivative we solve the following three diffusion equations by fully implicit methods:

\[ \frac{Dn}{Dt} = -\nabla \cdot \left( n \mathbf{v} \right) \]

\[ \frac{D\mathbf{v}}{Dt} = -\nabla \cdot \left( \mathbf{v} \mathbf{v} \right) \cdot \mathbf{v} \]

\[ \frac{D\mathbf{B}}{Dt} = -\nabla \cdot \left( \mathbf{B} \mathbf{v} \right) \]

The resistivity tensor \( \eta \) allows for anisotropy while the thermal flux \( \mathbf{J} \) and \( \mathbf{E} \) have the form \( \eta \mathbf{v} + \mathbf{J} \). The specific heat is \( \rho c_p = n m / (v+1) \) with \( v = 5/3 \). \( \mathbf{v} \) is the velocity of energy exchange between electrons and ions and \( \mathbf{Q} \) are the heat transfer rates to electrons and ions; these rates include ohmic heating, turbulent heat and bremsstrahlung. The transport of energy from the field to the plasma (\( \lambda = 2 \cdot \lambda \)) is calculated by a method which ensures exact energy conservation [6].

Initial distributions of n, T_e, T_i, B_r, B_θ and B_ψ have the form

\[ f(r) = f_0 + f_1 r/\lambda \]

and are then relaxed adiabatically in equilibrium before the calculation begins. The wall radius is 0.6m, and the values of \( f_0 \) and \( f_1 \) are

\[ n_{\text{wall}}^0 \Gamma T_e, T_i \quad B_r, B_θ, B_ψ \]

\[ \Gamma = 10^{-5} \quad 0.45 \quad 0.45 \quad 0.518 \]

These values correspond to a current \( i = 0.7 \) MA and a pinch parameter \( \alpha = 1.1 \). The calculations are carried forward in time at which \( B_r \) has increased to a value \( B_r > B_ψ \) so that the plasma becomes MHD ideal-unstable. The value of \( B_ψ \) depends on the details of the plasma profile but for simplicity \( B_ψ = 2.1 \) is used [7].

3. TURBULENT TRANSPORT

Although the linearized stability theory of the pin a pinch [1,6] and other MHD configurations has been studied in detail, what happens when an unstable mode grows into the non-linear regime is not well understood. Ideal MHD theory [3] describes the departure from equilibrium by a lagrangian displacement vector \( \mathbf{d} \) in which the linear regime satisfies the equation of motion

\[ \frac{d\mathbf{d}}{dt} = \mathbf{J} + \mathbf{d} \]

where the operator \( \mathbf{J} \) is given in [8]. As \( \lambda \) increases, points may occur at which the Jacobian \( \frac{d\mathbf{d}}{dt} \) of the transformation

\[ \mathbf{d} = \mathbf{u} + \mathbf{d} \]

becomes zero or undefined so that the transformation itself becomes singular. This may be expected to lead to arbitrarily steep magnetic field gradients and hence to a breakdown of the ideal MHD theory due to resistive effects however small the resistivity \( \eta \) may be.

In cylindrical geometry \( \lambda \) is resolved into linearly independent eigenmodes

\[ \lambda = \lambda_\parallel + \lambda_\perp \quad \lambda_\parallel = \frac{1}{2} \left( \frac{\lambda_\parallel}{\lambda_\perp} \right) \]

(8)

so that only the radial component \( \lambda_\parallel \) (denoted by \( \lambda \) for brevity) need be studied explicitly. The radial Jacobian is

\[ \frac{d\mathbf{d}}{dt} = 1 + \frac{\lambda}{\lambda_\perp} \]

(9)

so that the transformation becomes singular as \( \frac{\lambda}{\lambda_\perp} = 1 \) (the absolute value being appropriate since the sign of \( \lambda \) is arbitrary). We therefore expect each eigenmode to lead to a localized loss of containment at the radius where \( \frac{\lambda}{\lambda_\perp} \) takes its largest value. Elsewhere the transformation (7) still remains non-singular and therefore should preserve the topology of the magnetic surfaces.
The stability of an equilibrium is studied by minimizing the change in energy $\Delta W$. This implies $\nabla \cdot \delta = 0$ (11) and leads to

$$\Delta W = \int \frac{\partial U}{\partial \xi} \cdot \delta \, dV$$

with $\delta$ given in (11). The displacements $(\delta r)$ which minimize $\Delta W$ satisfy the Euler-Lagrange equation

$$\frac{d}{dr} \left( \frac{d\delta}{dr} \right) - \frac{\partial U}{\partial \xi} = 0$$

which is also the equation for $\xi$. A marginal mode is a solution (11) satisfying $d\xi = 0$ (except for $n = 1$ when $d\xi = 0$). At the singular surface $r_0$ defined by

$$k \mu B_0 r + \kappa B_0 = 0,$$

with $\mu = \frac{B_0}{\partial B_0}$ the pitch length, the Sudakov-necessary stability criterion (13) is obtained by expanding (11) around $r_0$.

$$A = \frac{\nu}{\nu} \left( r B_0 \frac{\partial B_0}{\partial r} \right)^2 = 1 = 0$$

The solution of (11) in the neighborhood of $r_0$ varies as $e^{(r-r_0)\lambda}$ (9), where $\delta$ is a constant and $\lambda = -\frac{\mu B_0}{\nu}$. If condition (15) is violated $\lambda > 0$ then $\delta$ is complex and $\lambda$ oscillates infinitely rapidly at $r_0$, i.e., is unstable. Further away from $r_0$ the solution $\delta$ decays towards zero at $r = 0$ and $r = \infty$ (for values of $\delta$ at $r = 1$). The solutions $\lambda$ arising at a violation of (15) have large $\lambda_0$ highly localized to the region near $r_0$ and are referred to as Sudakov modes or localized interchange modes. There may exist unstable solutions to (11) that are present even if (15) is not violated; such solutions represent non-localized modes (grad-nos). These are not considered in the present paper; their precise form depends on the details of the function $\phi$.

Localized unstable modes thus cause $\mu B_0$ to grow at $r_0$, corresponding to large radial compression of the magnetic surfaces. Since the plasma is resistive, magnetic field lines can break up thereby allowing plasma to move across, so leading to a localized anomalous plasma and thermal transport.

5. TRANSPORT COEFFICIENTS

The turbulent transport is described by a turbulent diffusion coefficient which we expect to be large, although its functional form is not known. Based on no experimental evidence (30,11) we let the turbulent diffusion coefficient scale with "the poloidal beta" coefficient

$$D_T = \frac{1}{\kappa \nu} \frac{B_0}{\nu}$$

To make the calculation more controllable we introduce a 'thermostat function'

$$\Theta_{\xi} = \int \Theta_{\xi} \, dV$$

which switches on the turbulent transport smoothly in the Sudakov-unstable zone where $\Delta > 0$. The results should be almost independent of the magnitude of $\Theta_{\xi}$ and the form of $\Theta_{\xi}$ (5) provided that $D_T$ is sufficiently large, and this is borne out by numerical calculations (16). The transverse resistivity and thermal conductivities are then defined as

$$\eta_\perp = \frac{\nu}{\nu} \left( \frac{\partial B_0}{\partial r} \right)^2 + \frac{\mu^2}{\nu^2}$$

$$\eta_\parallel = \frac{\nu}{\nu} \left( \frac{\partial B_0}{\partial z} \right)^2 + \frac{\mu^2}{\nu}$$

where subscript $\perp$ refers to classical values (Spitzer), and

$$\eta_\perp = \frac{\nu}{\nu} \left( \frac{\partial B_0}{\partial r} \right)^2 + \frac{\mu^2}{\nu^2}$$

The resistivity parallel to the magnetic field remains classical. The numerical parameters $\eta_\perp, \eta_\parallel$ can be adjusted to vary the turbulent transport level in the unstable zone, while $\eta_\perp, \eta_\parallel$ enable the effect of anomalous transport in the stable zone to be investigated.
Diffusive Transition from Belt Pinch to Doublet

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The Grad-Hogan classical diffusion model [1]
\[ \nabla p = J \times B \]

(1)
\[ \partial B / \partial t + \nabla \times (B \times u) = \nabla \times (\eta J) \]
\[ \partial p / \partial t + u \cdot \nabla p + \gamma \rho \text{div} u = 0 \]
drops the inertia term \( \rho \partial u / \partial t \) in order to save many orders of magnitude numerically. In exchange, it poses a very nonstandard system of equations, both analytically [2] and numerically [3]. For simplicity, heat flow is ignored (isothermally with \( \gamma = 1 \) and adiabatically with \( \gamma > 1 \)). This model includes coupled skin effect and "classical" plasma diffusion. Earlier "classical" models (Kruskal-Kulsrud, Pfirsch-Schluter, Neoclassical), based on the postulate, \( \nabla \times E = 0 \) or \( \partial B / \partial t = 0 \), describe only a limited special class of solutions of (1) [4]. In a low \( \beta \), large aspect Tokamak, the solutions decouple into fast (skin effect) and slow (classical) components; though the time scales decouple, both are governed by nonstandard diffusive systems. In a general, finite aspect or finite \( \beta \) case, the Pfirsch-Schluter (or Neoclassical) special profiles are not relevant.

Many of the significant analytic, numerical, and physical points are contained in the 2-D low \( \beta \) model
\[ \partial \psi / \partial t + u \cdot \nabla \psi = \eta \partial \psi \]

(2)
\[ \partial \psi = F(\psi, t), \quad <u \cdot \nabla \psi> = 0 \]
where
\[ F = -\partial p / \partial \psi - \int_{0}^{\psi} \frac{df}{d\psi}, \quad B_{2} = f_{0} + f(\psi) \]
\[ \psi \equiv \int_{S} dS / |V|, \quad \int_{S} dS / |V| = 1 \]

and \( V \) is the volume (area in 2-D) within a flux contour. On the relevant time scale, the mean flow \( <u \cdot \nabla \psi> \) is negligible and \( p(V) \) is independent of time. Plausible mathematical arguments show that a solution \( \psi(x,y,t), F(\psi, t) \) and \( u \cdot \nabla \psi \) of (1) is determined by the initial value \( F(\psi, 0) \) and a boundary value for \( \psi(x,y) \) [2]. Since \( p \) is known, \( F = B_{2} \)
can be found from \( F \). The higher order net diffusion velocity is given in the Grad-Hogan theory by the formula (\( K \) is defined below, \( U_{c} \) is classical)
\[ U = U_{c} - [K \psi'_{t} + f_{0} \int f_{t} dV] / B^{2} \]
\[ U_{c} = -\eta K \psi' / B^{2} \psi' \]

The analytic Green's function treatment of (1) in Ref. [2] is numerically impractical. A practical numerical algorithm [3] is obtained by alternately: (i) diffusing the profile \( \psi(V,t) \) in 1-D

(5)
\[ \psi_{t} = \eta (K \psi')', \quad K(V) = <|V|^{2}> = f(V, S) \]

where \( K \) is a geometric (inductance) coefficient; (ii) at intervals recalculating \( K(V,t) \) from the 2-D contours \( \psi(x,y) = \text{const} \). A complete analytical and numerical treatment of the analogous adiabatic problem has been given, including creation and development of islands (as in Doublet) using a generalization of the concept of adiabaticity to treat mixing and splitting of islands. More precisely, the algorithm is as follows:

- contours \( t_{0} \) \( \rightarrow \psi(V,t_{0}) \) \( \rightarrow \psi(V,t_{1}) \) \( \rightarrow \psi(x,y,t_{1}) \) \( \rightarrow \psi(x,y,t_{1}) + \text{contours}(t_{1}) \)

The geometric coefficient \( K(V) \) is determined by a given family of contours. A new set of contours is determined by Poisson inversion of \( \Delta \psi = F \), then taking \( \psi(x,y) = \text{const} \).

A general complex configuration requires connection formulas which will be discussed elsewhere. In a Doublet configuration (Fig. 3) with symmetric islands, a simple artifice allows use of exactly the same algorithm which suffices for the simple geometry. One merely doubles the value of \( V \) and quadruples the value of \( K \) assigned to a \( \psi \)-contour in an island. Since the renormalized \( \psi \) and \( K \psi' \) are continuous, the 1-D diffusion equation is solved uniformly from \( V = 0 \) to the edge of the plasma. The code senses when the topology changes in order to correctly calculate contours, \( V, K(V) \), etc.

For greater accuracy one should recognize that there is a singularity at the separatrix, viz. \( K \sim \log |V-V_{\text{GS}}|, \psi' \sim 1 / \log |V-V_{\text{GS}}| \); one can also make \( K(V,t) \) piecewise linear in \( t \) instead of a step function by iterating on \( K(t_{1}) \) over \( t_{0} < t < t_{1} \); some care is required in
transferring values among the coarse V mesh (contours), the fine V mesh (diffusion), and the 2-D mesh. Also, by suitable use of inductance in the vacuum, one can avoid computing any contours in the vacuum. An accurate method of calculating K in terms of the Poisson-inverted function \( \psi(x,y) \) is as the product of \( \phi ds/|\nabla \psi| = 1/\psi' \) and \( \int \nabla \psi |dS = K \psi' \). We point out that the moving separatrix introduces a boundary layer because of the moving singularity of \( K(V) \) (Fig. 4) and special care must be taken at the initial appearance of small islands.

A slight generalization of (2) allows toroidal compression, \( f_0(t) \),

\[
\psi_t + U \psi' = \eta(K \psi')
\]

\[
U(V,t) = a(t)V, \quad a = -f_0'(t)/f_0
\]

The result is an efficient code, (i.e. mostly 1-D with only occasional excursions into 2-D). There is a smooth transition from Belt Pinch to Doublet, with three arbitrarily given inputs: plasma current \( I_1(t) \), shaping coil current \( I_2(t) \), and toroidal field \( f_0(t) \). Current boundary layers are observed at the separatrix and the plasma edge.

This code has many features in common with the earlier adiabatic codes[5] which include islation through a generalization of the concept of adiabaticity. The principal difference lies in an entirely different singularity at the separatrix and entirely different connection formulas. For example, \( K = \log|V-V_S| \) for diffusion, while \( K = 1 \) but \( K = V^{-1/3} \) (on the acute angle side) adiabatically.

This code can be extended (but only by careful preliminary analytical preparation and selection of variables and algorithms) to axial symmetry at finite aspect and finite \( \delta \), including Diverter and tearing as well as Doublet configurations without significant loss of efficiency (to appear).

\[\text{References}\]


A general approach to solving the rate equations for ionisation-
combination processes in a plasma

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The aim of the present work is to develop computer models for the atomic
physics processes in a partially ionised time evolving plasma. Such models
must be efficient and accurate and are intended to supplement an MHD
description of the plasma in computer codes.

Instead of solving for the ionisation fractions directly, the method
used here solves for the moments of the distribution function $f_\nu$ of the
ionisation stages, namely

$$<x_\nu^m> = \frac{1}{\mu = 1} \mu^m f_\mu$$

These moments $<x>, <x^2>$ etc. are the quantities of interest for evaluating
the electron density, radiation rates etc.

We can obtain from the rate equations a set of coupled non-linear
ordinary differential equations for the moments of the distribution function
defined above. Only a few of these moments need then be considered to
obtain an adequate description of the plasma. The accuracy of the model at
any stage in the calculation depends only on the number of moments included,
and will be improved upon by including higher order moments. Results from
a number of these models will be presented together with a discussion of
their accuracy and computational efficiency.

1. INTRODUCTION

The rate equations governing the ionisation-combination processes in a
plasma are usually written in the form:

$$\frac{df}{dt} = \begin{cases}
-\nu f_\nu + \nu f_\nu & \text{if } \mu = 0 \\
-\nu \Sigma_{\mu=1}^{\nu-1} f_{\mu} + \nu f_\nu & \text{if } \nu < Z \\
-\nu \Sigma_{\mu=1}^{Z} f_{\mu} + \nu f_\nu & \text{if } \nu = Z 
\end{cases}$$

where $\Sigma$ and $\nu$ are the ionisation and recombination coefficients\(^{(1)}\) and $f$ is
the ionisation fraction for $\nu$-times ionised atoms. This set of equations
can be replaced by the matrix equation

$$\frac{df}{dt} = [A_{\nu \nu}] f_\nu$$

$A_{\nu \nu}$ is now a matrix of rate coefficients. Previous authors\(^{(2)}\) have solved
the set of equations (1.2) more or less directly to obtain an estimate of the
degree of ionisation in the plasma. Instead of solving for the ionisation
fractions, the approach here is to solve for the moments of the distribution
function $f_\nu$, defined by

$$<x_\nu^m> = \frac{1}{\mu = 1} \mu^m f_\mu$$

These moments are in fact the quantities of interest (rather than the
fractions) since from $<x>, <x^2>$ etc. we can obtain the electron density,
radiation rates etc. Also since $<x>, <x^2>$ etc. are much smoother varying
functions of time (cf. the fractions) we should expect solving for the $<x>$
to be less computationally severe.

2. THE MODEL

To obtain the transformation from the $f_\nu$ to $<x_\nu>$ is quite simple.
Multiplying equation (1.2) by $\mu^n$ and summing over $\mu$ we have

$$\frac{d}{dt} <x_\nu^m> = \nu f_\nu + \nu \Sigma_{\mu=1}^{\nu} A_{\mu \nu} <x_\mu^m>$$

or

$$\frac{d}{dt} <x_\nu^m> = \nu <x_\nu^m>$$

for $\nu = 1, 2, ..., Z$  (2.1)

where $A_{\mu \nu} = \frac{1}{\nu + 1} - \frac{1}{\nu} \Sigma_{\mu=1}^{\nu} A_{\mu \nu}$ $A_{\nu \nu} = \frac{1}{\nu} - \frac{1}{\nu} \Sigma_{\mu=1}^{\nu} A_{\mu \nu}$. Although the right
hand sides of (2.1) still contain the $f_\nu$ these can be replaced by expressions
involving the $<x_\nu^m>$. From (1.3) solving for the $f_\nu$ gives $f = f(<x>,<x^2>, \ldots, <x^Z>)$ and the system (2.1) reduces to

$$\frac{d}{dt} <x_\nu^m> = \nu <x_\nu^m>$$

(2.2)

where $\nu$ is the ion density and we have used $\nu = <x>_1$ in (2.2).

Using this approach we can develop some simple models.

a. Assuming only two fractions present at any time (reasonably populated)
are, e.g. $f_1$ and $f_2$, i.e. equation (2.1) gives for $\nu = 1$

$$\frac{d}{dt} <x_\nu^m> = \nu (S_1 f_1 - R_1 f_2)$$

(2.3)

Also since $f_1 + f_2 = 1$ and $<x_\nu^m> = x f_\nu + (x - 1) f_\nu + 1$ we can rewrite (2.3),
using $\nu = <x>_2 f_2$, as

$$\frac{d}{dt} <x_\nu^m> = A <x_\nu^m> + B$$

with solution

$$<x_\nu^m> = C(1 - AC)^n$$

(2.4)

where $<x>_0$ is the initial value of $<x>$.  

b. Extending the calculation to three ionisation fractions, we must know
$<x>, <x^2>$ to specify the system completely. However under the
assumption $<x^3> = <x^2>$ and holding the electron density constant over
the timescale $t$, (2.1) leads to the equation

$$\frac{d}{dt} <x_\nu^m> = A <x_\nu^m> + B <x_\nu^m> + C$$

(2.5)

which written in the form

$$\frac{d}{dt} <x_\nu^m> = k_2 <x_\nu^m> + k_1 <x_\nu^m> + k_0$$

has the standard solution

$$<x_\nu^m> = \frac{k_2}{k_1} A \cosh k_1 t + \frac{1}{k_1} \left[ A + \frac{1}{4} B \right]$$

(2.6)
In the more general situation we cannot assume \( e^2 = e^3 \) and the above simple solutions do not apply. Still using the three fraction approximation, equation (1.3) gives, on solving for the fractions, and denoting \( f(\alpha^*) \) by \( \mu \),

\[
\begin{align*}
\mu_1 &= 0.5 \left[ (-2 \mu + 1) \alpha^2 + \alpha^2 + \mu (\mu + 1) \right] \\
\mu_2 &= 2 \mu \alpha^2 - \alpha^2 - (\mu + 1) (\mu - 1) \\
\mu_3 &= 0.5 \left[ (1 - 2 \mu) \alpha^2 + \alpha^2 + \mu (\mu - 1) \right] 
\end{align*}
\tag{2.7}
\]

which upon substitution into the equations obtained from (2.1) for \( n = 1, 2 \), i.e.

\[
\begin{align*}
\frac{dn}{dt} &= \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_1 \right] \\
\frac{dn}{dt} &= \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_2 \right] \\
\frac{dn}{dt} &= \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_3 \right] 
\end{align*}
\tag{2.8}
\]

gives the equations

\[
\begin{align*}
\frac{dn}{dt} &= \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_1 \right] \\
\frac{dn}{dt} &= \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_2 \right] \\
\frac{dn}{dt} &= \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_3 \right] 
\end{align*}
\tag{2.9}
\]

and the system of equations (2.9) is solved implicitly. This approach forms the basis of an atomic physics model for use in computer codes. It should be pointed out that although this model is equivalent to solving for the three fractions \( \mu_1, \mu_2, \mu_3 \), as discussed in a previous page (9), it is a more efficient algorithm. The disadvantages in solving for the \( e^2 \) become apparent when we include the calculation for higher order moments. Essentially we are solving

\[
\frac{dn}{dt} = \frac{n}{2} \left[ (S_{\mu} - E_{\mu}) \mu_1 \right] 
\tag{2.10}
\]

for the \( e^2 \) and the matrix \( G_{\mu} \) has no symmetry properties (cf. equation (1.2)) where \( G_{\mu} \) is tridiagonal. Thus as we include the calculation for \( e^3 \) and higher order moments, the algebra becomes increasingly more complicated and soon reaches the point that it would be more efficient to solve for the \( f_\mu \) (exploiting the tridiagonal properties of \( A_{\mu} \)).

**RESULTS**

We applied the three fraction model of section 2c to a) an ionising and b) a recombinating plasma whose temperature and electron density were held constant and the results were compared with the exact solutions (3) in Figs. 1 and 3. We obtain good agreement for the ionisation but not so good during recombination, as has been pointed out earlier (2). Figure 4 shows the exact recombination results and we observe that a three fraction approximation is not too good in this situation. Figure 4 shows a complete ionisation recombination calculation (allowing the electron density to vary) for a square wave temperature pulse. In each of the results with the three fraction model we have found that by adjusting the degree of implicitness equal to 1 and \( \frac{1}{2} \) gave indistinguishable results. Also the recombination coefficients (being density dependent) were evaluated at \( n = 10^2 \) cm\(^{-3}\) for simplicity. Extending the calculation to a four fraction model (now we solve for \( e^3, e^4, e^5, e^6 \)) and using \( s = 1 \) gave improved results in the recombination test.

**CONCLUSION**

Instead of solving for the ionisation fractions directly we solve for the \( \alpha^2, \alpha^3 \), etc., which are the quantities of interest. Under the assumption of only three adjacent ionisation fractions being reasonably populated at any time a model has been developed to follow the ionisation and recombination of the plasma. In the case of rapid recombination the three fraction approximation is not necessarily a good one and extension to a four fraction model gives improved results, although the algebra becomes considerably more complicated. For a higher accuracy than that provided by the four fraction model, we suggest reverting to solving directly for the ionisation fractions exploiting the tridiagonal nature of the matrix \( A_{\mu} \) in (1.2).
A SIMPLE ROUTINE FOR REACTION PRODUCTS TRANSPORT
IN LASER-PRODUCED PLASMA
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The deposition of energy from nuclear reaction products in a fusing D-T plasma ignited by a high powered laser, leads to the onset of a burn wave which propagates outward from the center of the pellet. The main energy carriers from the burning fuel into the adjacent colder regions are the alpha particles. The contribution of protons from D-D reactions is about two orders of magnitude smaller and the neutron energy deposition is of importance only for very high densities of the order of 1000 g/cm³ or higher.

A simple trace tracking routine for charged particle reaction products has been developed as part of a one-dimensional spherical hydrodynamic laser plasma simulation code(1). This routine calculates the spatial dependence of the energy deposition inside the pellet and the energy spectrum of those particles which escape. Recent measurements have attempted to measure the alpha particle energy spectrum in order to determine if these are generated in the core or in the outer corona(2).

The spherical plasma volume is divided into concentric homogeneous shells. Since nuclear reactions take place uniformly in each shell, an approximate representation assumes that they all occur at a radius, R, which divides the shell into two equal masses. It can be shown by direct integration that this approximation best matches the exact solution. From each generation point, reaction products are assumed to be emitted isotropically and the 4π solid angle is divided into cones such that an equal number of particles is emitted into each cone. Since slowing down by electrons dominates along most of the charged particle range, these particles are assumed to move along straight lines. The distance, D, travelled by the fast particles in each shell is calculated from the geometry of the trace. Outgoing and ingoing trajectories are treated separately.

Consider a fast ion of mass, M, and charge, Ze, moving with a velocity, v, in a plasma consisting of electrons and ions (of charge Z₁e). The electrons and ions have thermal velocities, ωₑ and ω₁, number densities, nₑ and n₁, and masses, mₑ and m₁, respectively. The rate of energy loss to the plasma due to Coulomb interactions is given by(3):

$$\frac{dE}{dx} = -\frac{4N}{\pi} \left( \frac{Z_e}{\omega_1} \right)^2 \ln \left( \frac{\sigma \omega_1}{\omega_1} \right) \left( \frac{v}{\omega_1} \right) + \frac{\rho \sigma}{m_e} \left( \frac{v}{\omega_1} \right)^2$$ \hspace{1cm} (1)

where the first and second terms are due to slowing down by ions and electrons respectively. G(y) is given by:

$$G(y) = \int_{0}^{y} \exp(-y^2)dy(y(1 + \frac{M}{\rho}) \exp(-y^2)$$ \hspace{1cm} (2)

The two limiting values of G(y) are:

$$-\frac{M}{\rho} y + \left( \frac{3}{2} \pi \frac{M}{\rho} \right) y^3 \hspace{1cm} \text{for} \hspace{0.5cm} y \ll 1$$ \hspace{1cm} (3)

and

$$\frac{1}{2} \left( \frac{M}{\rho} \right) \hspace{1cm} \text{for} \hspace{0.5cm} y \gg 1$$

For the reaction products and plasma temperatures of interest here, \(\frac{M}{\rho} \gg 1\) (see Fig. 1). The slowing down rate by ions is, therefore, taken to be,

$$\frac{dE}{dx}_\text{ions} = -\frac{2z_e^2 (M)}{m_1} \left( \frac{2\pi}{\omega_1} \right)^2 \ln \left( \frac{\sigma \omega_1}{\omega_1} \right) \frac{v}{\omega_1}$$ \hspace{1cm} (4)

while the slowing down rate by electrons is taken from Eq. (1), with G(y) stored in the computer program in tabular form.

The routine can track each of the charged reaction product species by means of an external input. Those particles which are not transported are presumed to give their energy to the electrons in the generating shell. When the transport code is applied, the energy deposited in each shell is calculated. Together with the charged particle, Z electrons leave the generating shell. Their thermal energy is subtracted from the generating shell, and an amount equal to the difference in electron energy between
the adjacent shell \( I \pm 1 \) and \( I \) is added to each shell, \( I \), along the track. These electrons are added to the final shell along the trajectory to preserve charge neutrality. The final energy balance is then converted into electron and ion temperature changes. The number of electrons and ions is also updated at the end of each cycle.

The code assumes that each of the reaction products is thermalized (or escapes from the pellet) in a single time step. This assumption, which greatly simplifies the "bookkeeping," is usually not entirely correct. However, it is acceptable if the plasma does not undergo drastic changes on a time scale comparable with the particle lifetime inside the pellet (of the order of 10 picoseconds for an alpha particle in a 200 \( \mu m \) pellet).

In calculating the energy deposition and emission spectra, the spread in collision energy due to the thermal velocity distribution of the reactants, and range straggling of the products, were ignored. These two effects have to be taken into account only when one is interested in calculating the emission spectrum from a plasma which has a small central hot core as the source of the nuclear reactions, surrounded by an absorbing material. When dealing with an extended volume source of alphas, however, the main contribution to the energy spread comes rather from the range of track lengths within the plasma.

In order to test the validity of the method, and to find the optimum division of the pellet into shells and angular groups, the simple case of an homogeneous sphere of D-T was treated analytically. The analytic treatment for this comparison assumes the same physical model as in this numerical case, namely, that the particles travel in straight lines and are slowed down only by electrons. If one takes the \( \frac{3}{2} \gamma^3 \) approximation for \( G(y) \), one can integrate directly over radius and angle and obtain the following spectrum of emitted particles:

\[
dn/dE = \frac{2E}{3} \left( \frac{a^2 R^2}{E} \right)^{\frac{3}{2}} \left( \frac{\sqrt{E}}{E} - \frac{\sqrt{E}}{E_0} \right)^2 \tag{5}
\]

where \( \delta \) is the number of particles produced in 1 \( cm^3 \), \( R \) is the radius of the pellet and \( a \) appears in the slowing down equation.

\[
\frac{dx}{dx} = -\alpha \sqrt{E} \tag{6}
\]

These results were compared with the results of the numerical calculations. The dependence of \( \gamma \), the RMS difference between the two spectra, on the number of shells and angular groups is shown in Fig. 2 for a typical case.

In conclusion, it is important to note that when \( (E/\gamma E) \leq 10^3 \), (see Fig. 1), the \( \frac{3}{2} \gamma^3 \) approximation to \( G(y) \) is a very "good" one and the energy loss in each shell can be calculated exactly. Above this range, the exact \( G(y) \) has to be used and the energy loss in each shell can be calculated to various orders of the difference scheme in energy.

A more fruitful approach may be to produce solutions to (1)-(3) under conditions so simplified that solution is possible, yet sophisticated enough that understanding of the basic physical processes underlying convection is achieved. This latter approach is the one that Dr. Weiss and I have adopted to attack this problem.

To this end we considered the following model problem. Confining an unbounded layer of fluid between two parallel horizontal plates in a constant downward gravitational field. Heat this layer from below and cool from above. Let these plates exert no viscous stress upon the fluid. Consider the fluid to be incompressible except in the buoyancy term where the density fluctuates solely as a function of the temperature, and assume that the physical properties of the fluid are independent of the thermodynamic state and constant (the Boussinesq approximation). Equations (1)-(3) can be reduced to a simpler set:

\[ \vec{\omega} = \vec{V} \times \vec{u} = -\nabla^2 \vec{Y}, \]  
\[ \vec{\rho} \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\vec{V} \cdot \vec{P} + \vec{S} + \nabla \varphi, \]  
\[ \vec{\rho} \left( \frac{\partial \vec{T}}{\partial t} + \vec{u} \cdot \nabla \vec{T} \right) = \vec{S} \cdot \nabla \vec{V} + \vec{V} \cdot (\vec{K} \vec{V}). \]

where \( \vec{\omega} \) is the vorticity and \( \vec{Y} \) a stream function. When all of the dependent quantities are functions of just two space variables, the vorticity and the stream function reduce to single component vectors and equation (5) can be written for scalar \( \vec{\omega} \) and \( \vec{Y} \).

We solve this system by assuming the motion to be periodic in the horizontal direction. We cover a region in 2 dimensional coordinate space with a regular grid of points. The differential operators in (4)-(6) are represented by 2nd order space and time centered difference operators. The resulting difference equations are solved at discreet moments in time. A solution starts from some initial set of fields and evolves in time until a steady state is achieved or the motion is followed for a sufficient span of time to document the behavior fully.

The elliptic equation (4) is solved exactly at each time step by variants of the method first described by Hockney (1965) suitable for the staggered mesh and coordinate system of the problem. The parabolic equations (5) and (6) are approximated as suggested in Roberts and Weiss (1966) in an integral conservative form. Exact details of the implementation for this scheme for a cartesian mesh are to be found in Moore, Peckover and Weiss (1973) and for a cylindrical mesh in Jones, Moore and Weiss (1976).

Our first model problem was that of Rayleigh- Benard convection in 2 dimensional cartesian geometry (Moore and Weiss (1973)). There we found a heat transport law

\[ N \sim h^{2.53 - 2.37}. \]
where $N$, the Nusselt number, measures the ratio of the total heat transported to the heat that would be conducted in the absence of motion, and $R_e$, the Rayleigh number, measures the ratio of buoyant potential energy to dissipation by conduction and viscosity across the fluid layer. The range of the exponents for $R_e$ correspond to $p$, the ratio of viscosity to conductivity, varying from $p = 0.1$ to $p = 0$. These two limits are relevant to the interior of the earth ($p = 0$) and to stellar interiors ($p = 0$). The efficiency of convection in this highly idealized model is seen to depend weakly upon $p$ and in the sense that $N$ increases as $p$ decreases for a fixed value of $R_e$.

It has been suggested on the basis of a simple finite amplitude expansion (Malkus and Veronis, 1955) that if three dimensional motion were considered, then $N$ would depend directly upon $p$ and would decrease as $p$ decreases. As convection in the Sun is observed to be of three dimensional character, this discrepancy between two dimensional and three dimensional heat transport in the astrophysical system should be resolved.

Axisymmetric geometry was considered as a two-dimensional realization containing the essence of the relevant three dimensional effect. Finite amplitude analysis of this geometry, Lioig, Vidat, and Arrivos (1967) had suggested the same behavior for $N$ upon $p$ in this case as in the 3-D case.

In the axisymmetric model, equations (4)-(5) take the form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \omega}{\partial r} \right) = -\frac{\partial \omega}{\partial z}$$

(8)

$$\frac{\partial \omega}{\partial t} = \frac{1}{r^2} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial \omega}{\partial r} \right) - \frac{\partial \omega}{\partial z} \right)$$

(9)

$$\gamma = \frac{1}{r^2} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial \omega}{\partial r} \right) - \frac{\partial \omega}{\partial z} \right) + \frac{\partial \omega}{\partial z}$$

(10)

These were solved by the methods used in the cartesian work. Full results are available in Jones, et al., but some interesting results will be discussed here.

For values of $R_e$ very near the critical value necessary for the onset of motion, the behavior is exactly as predicted by the finite amplitude analysis. However, as $R_e$ is increased, a surprising thing occurs. The convection quickly becomes much more efficient until for $R_e$ twice the critical value it is 50% as efficient for the smallest value of $p$ investigated as for the largest. This behavior is qualitatively very different from that predicted by the finite amplitude analysis. By the time the Rayleigh number is 20, the critical value small $p$ convection is more efficient than large $p$ convection, a complete reversal of the analytical prediction.

Figure 1 reveals how the convection manages to be so efficient in this geometry when the layer is very unstable and viscosity is much less important than conductivity. In this case a grid with 40 vertical and 50 horizontal intervals covers one cell.

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LOW PRANDTL NUMBER THERMAL CONVECTION

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1. INTRODUCTION. Low Prandtl number convection calculations can be carried out using only one evolutionary equation, and solving Poisson’s equation twice. A boundary condition on the vorticity equivalent to a no-slip condition has been constructed for explicit evolutionary finite difference schemes.

When \( \sigma \gg 1 \), the advective terms in the momentum equation can be ignored. This simplification has been exploited by Busse (e.g. Busse and Whitehead 1971) by Elder (1965a) in the vertical slot problem, and by McKenzie, Roberts and Weiss (1973) in calculations relating to motions in the earth’s mantle. In the next section we indicate simplifications which arise when \( \sigma \ll 1 \), appropriate for liquid metals.

In astrophysics, a no stress boundary condition may be no worse an approximation to the actual state of affairs than a no-slip condition. In engineering applications the correct form for the boundary layer, which determines the magnitude of the thermal fluxes is essential. In section 3, we give, following Lighthill (1963), a no-slip boundary condition appropriate for a vorticity-stream function formulation on a staggered mesh.

2. LOW PRANDTL NUMBER FORMULATION. For thermal convection in a Boussinesq fluid confined to a rectangular region in the \( x-z \) plane, the flow can be described by the temperature \( T \), the vorticity \( \omega \), and stream function \( \psi \). With the units indicated below, these satisfy the equations

\[
\begin{align*}
\frac{\partial T}{\partial t} &= -\mathbf{v} \cdot \nabla T + \nu \nabla^2 T \\
\frac{\partial \omega}{\partial t} &= -\mathbf{v} \cdot \nabla \omega + \nu \nabla^2 \omega - \frac{\partial T}{\partial x} \\
\frac{\partial \psi}{\partial x} &= -u
\end{align*}
\]

(1)

where \( \mathbf{v} = (u,0,\omega) = (-\omega/2,0,\nu/\kappa) \) and \( \kappa \) and \( \nu \) are dimensionless thermal and viscous diffusivities. These non-dimensional equations express lengths in fractions of the layer depth \( L \), the time in multiples of the Brunt-Väisälä period \( T_B \) and temperature differences in multiples of the total temperature drop.

The set of equations (1) can be solved by finite difference methods. One method using the leapfrog and Dufort-Frankel scheme (Roberts and Weiss 1966), has been implemented on a staggered mesh as the program GZ1OTA (Moore, Peckover and Weiss 1973) and used in studies of Bénard convection (Moore and Weiss 1973), convection with internal heat sources (Peckover and Hutchinson 1974), convection in the earth’s mantle (McKenzie, Roberts and Weiss 1973) and convection in a magnetic field (Peckover 1969, Peckover and Weiss 1972).

The Dufort-Frankel approximation to the diffusive term is unconditionally stable but accurate only if \( \nu \partial T/\partial x^2 \ll 1 \), \( \kappa \partial \omega/\partial x^2 \ll 1 \). For computer efficiency, the timestep must be kept as large as possible. For \( \sigma \ll 1 \) (for liquid sodium \( \sigma \approx 0.005 \), for the solar convection zone \( \sigma \approx 3 \times 10^{-9} \) using a radiative conductivity (Weiss 1964)) the size of the timestep is controlled by \( \nu \), which is several orders of magnitude more restrictive than either the viscous constraint, or the CFL condition.

For low Prandtl number, on the viscous timescale, the relaxation of the temperature field through thermal conduction is essentially instantaneous. In the regime where the Peclet number \( (UL/\nu) \) is small even though the Reynolds number \( (UL/\nu) \) is large (so that \( \sigma \ll 1 \)) then the deviation of the temperature field from the purely conductive solution \( T_0 \) is small. Hence we may expand the temperature field in \( \sigma \) as a small parameter

\[
T = T_0(x,z) + \frac{\partial}{\partial \sigma} (\sigma \phi) \phi(\sigma)
\]

(3)

with \( (\partial \phi/\partial T_0) \approx o(\sigma) \). \( T_0 \) is a potential function; thus neglecting \( o(\sigma) \) we can write equation (1a) as

\[
\nu \frac{\partial \phi}{\partial \sigma} = -\frac{\nu}{\sigma} \frac{\partial}{\partial x} (\partial T/\partial x)
\]

(4)

This is a Poisson equation for \( \phi \), in terms of a source term involving \( T_0 \) and the current value of \( u \). \( \phi \) vanishes on the boundaries, and so equation (4) is identical in form and boundary conditions with (1c). The Poisson solver can thus be used twice.

Particular problems for which this approach is useful are (i) Bénard:

\[
T_0 = 1 - \sigma; \quad \text{vertical slot: } T_0 = x; \quad \text{stabilized slot: } T_0 = cx + (1 - cx).
\]

Problems (i) and (ii) are special cases of (iii) for which \( \nu \partial \phi / \sigma \partial x = -\kappa \partial^2 u / \partial x^2 \).

We have carried out calculations for liquid sodium in the vertical slot problem using (6) instead of (1a) at each timestep, and obtaining the conductively relaxed temperature from \( T(t) = T_0 + \phi(t) \). Figure 1 shows the temperature and stream function profiles when the Rayleigh number \( R = 10^3 \) (using the gap separation as the length scale), \( \nu = 5 \times 10^{-5} \) and the aspect ratio was 2.5. The number of mesh points \( (N_z = 48) \) is sufficient to resolve the vertical momentum boundary layers. The method is a factor of \( \sigma^{-1} \) faster than the full GZ1OTA scheme.

3. NO-SLIP BOUNDARY CONDITION FOR THE VORTICITY-STREAM FUNCTION FORMULATION. The equations of thermal convection when cast in the form of (1) are well suited to the no stress, closed streamline hydrodynamic boundary condition. The no-slip condition \( u = w = 0 \) cannot be applied to a straight forward fashion. We give below explicit forms for the boundary vorticity in terms of interior quantities.

By making the rigid boundary AB (see figure 2) a streamline we satisfy \( u = 0 \) straightforwardly. The elliptic Poisson equation (1c) only requires one boundary condition, in this case \( \phi = \text{const} \). Thus \( w = 0 \) must be applied to
the solution of (1b). The practical problem in a computation is how to calculate $\omega$ on the boundary at each new timestep.

An explicit scheme has an important advantage over implicit ones in this regard, for the vorticity in the interior of the region can be calculated at a new timestep without the vorticity on the boundary being specified. Only the vorticity at previous timesteps is required. On the boundary, the point 0 is that for which we wish to calculate the vorticity at the $n^{th}$ timestep: $\omega_0^n$.

The stream function and vorticity at neighbouring points are $\psi_n^n$, $\omega_n^n$ etc (see figure 2 for notation).

From Taylor expansions for $\psi$ and $\omega$, we can write, after eliminating $r^{o}_{o_{xxx}}$ and using $\psi_o = 0$, and $h = \Delta x$,

$$
\psi_{o,x} = \frac{1}{6h} \left[ 8 \psi_1 - \psi_2 + 2 h^2 \omega_o \right] + \frac{1}{18} h^3 \left( \psi_{o,xxx} \right) .
$$

(8)

The no-slip boundary condition thus can be expressed as

$$
\left[ \begin{array}{c} \omega_o^n - \frac{1}{2h^2} \left( 8 \psi_o^n - \psi_2^n \right) \\
\end{array} \right] = 0(\Delta x^3).
$$

(9)

The error in the velocity can be written in terms of grid quantities as

$$
\frac{h}{2h^2} \left[ \begin{array}{c} \omega_o + \omega_2 - 2\omega_1 \\
\end{array} \right] = 0(\Delta x^3).$

This method of calculating the vorticity on the boundary is appropriate for explicit schemes. For the staggered leapfrog mesh used in C210TA, $\psi_1$ is interpolated from

$$
\psi_1 = \frac{1}{2} \left[ \psi_{n1} + \psi_{s1} + \psi_o + \psi_2 + \frac{1}{4} h^2 (\psi_{n1} + \psi_{s1} + \psi_o + \psi_2) \right]
$$

(10)

with fourth order accuracy (Moore, Peckover, Weiss 1973). This leads to

$$
\omega_o^n - \frac{1}{3} \omega_{n1} + \frac{1}{3} \omega_{s1} + \frac{1}{3} \omega_2 - \frac{2}{3h^2} \left( \psi_2 + \frac{1}{2} \psi_{n1} + \frac{1}{2} \psi_{s1} \right)
$$

(11)

which still satisfies the boundary condition $\omega = 0$ to $O(\Delta x^3)$.

An alternative formulation (e.g., Kawaguti 1961), Elder (1966) for the boundary condition combines the orthodox 5-point formula for $\psi_2$ with the symmetry condition $\psi_1 = \psi_0$ to give

$$
\omega_o^n = - \frac{2}{h^2} \psi_1^n
$$

(12)

The version of this appropriate for a staggered mesh is

$$
\omega_o^n = - \frac{1}{h^2} \left( \psi_{n1} + \psi_{s1} \right)
$$

(13)

These are lower order formulae; however since the equivalent of (8) for the Kawaguti formulation is

$$
\psi_{o,x} = \frac{1}{2h} \left[ 2 \psi_1 + h^2 \omega_o \right] + \frac{h^3}{12} \left( \psi_{o,xxx} \right) + 0(h^3)
$$

(14)

where $(\psi_{o,xxx})$ being $\frac{3h^3}{2h^3}$ on the boundary, is large.

These forms of the no-slip condition have been used to calculate heat transfer rates through a vertical slot filled with water with air. The results compare well with those of Elder (1965b) and of Eckert and Carlson (1961).

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The Application of Fast Fourier Transforms to the
Primitive Equations of Boussinesq Convection

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Introduction

There are a wide variety of numerical codes for solving the incompressible Navier-Stokes' equations. The staggered mesh code of Harlow and Welch (1965) deals with the equations in their primitive form, i.e. solves directly for the pressure field and the components of fluid velocity. It has the advantage of "compact" finite differences which appear to make it significantly more accurate than other second-order (space) schemes. (Orszag, 1971). The energy equation of thermal convection and the buoyancy force are easily incorporated into their scheme. Written in non-dimensional form the equations of the Boussinesq approximation to two-dimensional convection in an incompressible fluid are

\[ \frac{\partial \mathbf{u}}{\partial t} = \mathbf{f} - \frac{\partial}{\partial x}\left(\nu \frac{\partial \mathbf{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial \mathbf{u}}{\partial y}\right) - \frac{\partial}{\partial z}\left(\nu \frac{\partial \mathbf{u}}{\partial z}\right) + \mathbf{g} + \mathbf{e}, \quad (1) \]

\[ \frac{\partial \mathbf{e}}{\partial t} = \frac{\partial}{\partial x}\left(\nu \frac{\partial \mathbf{e}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial \mathbf{e}}{\partial y}\right) + \mathbf{g} + \mathbf{e}, \quad (2) \]

\[ \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial x}\left(\nu \frac{\partial \mathbf{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial \mathbf{u}}{\partial y}\right) + \mathbf{g} + \mathbf{e}, \quad (3) \]

\[ \frac{\partial \mathbf{e}}{\partial t} = \mathbf{0}. \quad (4) \]

Here \( \mathbf{u} \) and \( \mathbf{e} \) are the vertical and horizontal components of velocity respectively, \( \nu \) is the Prandtl number. The original form of Harlow and Welch's numerical scheme suffers from two main disadvantages, (i) the Poisson-type equation for the pressure field derived from equations (1), (2) and (4) is solved by iterative techniques, (ii) it is an explicit scheme and so required that the timestep satisfy

\[ \Delta t < \text{minimum}\left(\frac{\nu \Delta x^2}{(\Delta y^2 + \Delta z^2)}\right). \]

where \( \Delta y \) and \( \Delta z \) are the spacesteps in the \( y \) and \( z \) directions respectively.

The disadvantage of solving for the pressure field by iteration is that compressibility is introduced into the flow and this must then be compensated for by having additional terms on the right-hand-side of the pressure equation. The use of Fast Fourier Transforms is an efficient way of obtaining an exact solution to the pressure field, as many authors have pointed out. This has enabled us to calculate accurately incompressible flows without the use of correction terms.

The second disadvantage is a major one in that for a grid of \( N \times N \) points, the total time for computing any given flow will vary as \( N^4 \).

The source of the time-step restriction is in the diffusive terms of equations (1) - (3), and so it was decided to look for some way of calculating them implicitly. A well known method for solving the diffusion equation in 2 or 3 dimensions is the Alternating-Direction-Implicit technique of Douglas-Rachford (Mitchell, p.50). We decided to look for a straightforward way of extending this technique to equations (1) - (3), given the flexibility of an efficient direct solution of the pressure equation.

Details of the Numerical Solution.

The A.D.I. technique was adapted to the Boussinesq equations by simply collecting the non-diffusive terms together into a non-linear contribution \( \mathbf{f} \).

This gives a set of equations of the form

\[ \begin{align*}
(1 - \nu_2 \Delta z^2) \mathbf{u}^{n+1} &= (1 + \nu_2 \Delta z^2) \mathbf{u}^n + \frac{\nu}{\Delta z^2} \mathbf{G}\left(\mathbf{u}^n\right), \quad (5) \\
(1 - \nu_1 \Delta x^2) \mathbf{u}^{n+1} &= (1 + \nu_1 \Delta x^2) \mathbf{u}^n + \frac{\nu}{\Delta x^2} \mathbf{G}\left(\mathbf{u}^n\right), \quad (6)
\end{align*} \]

where \( \nu = \Delta t / \Delta \xi \sigma / \Delta z^2, \quad s = \Delta t / \Delta \xi \sigma / \Delta z^2, \quad u = \mathbf{G}\left(\mathbf{u}^n\right). \)

Equation (5) is explicit and the intermediate values \( \mathbf{u}^{n+1} \) can be calculated directly by tri-diagonal elimination. Equation (6) is implicit as it stands, so \( \mathbf{u}^{n+1} \) was calculated by using \( \mathbf{G}\left(\mathbf{u}^{n+1}\right) \) as a first guess for \( \mathbf{G}\left(\mathbf{u}^{n+1}\right) \) and then writing the equation in an iterative form. If we denote the \( i \)th iteration to \( \mathbf{u}^{n+1} \) as \( \mathbf{u}_i^{n+1} \), then we write

\[ \begin{align*}
(1 - \nu_2 \Delta z^2) \mathbf{u}_i^{n+1} &= (1 + \nu_2 \Delta z^2) \mathbf{u}_i^n + \frac{\nu}{\Delta z^2} \mathbf{G}\left(\mathbf{u}_{i-1}^{n+1}\right), \quad (7)
\end{align*} \]
The pressure equation is solved once to find \( P(\vec{u}^n) \), and successively to find \( P(\vec{u}^{n+1^*}), \ldots, P(\vec{u}^{n+1}) \). This ensures that the flow is accurately incompressible at every stage of the calculation. Equation (7) converged satisfactorily for the timesteps used in the calculations detailed below.

Numerical Results

This adapted A.D.I. technique was tested by computing the Nusselt numbers for convective steady-state solutions of the Boussinesq equations and comparing them with published values. These flows are for rigid stress-free boundaries and have an aspect ratio of \( 1:1/2 \). The Rayleigh numbers quoted are in terms of the critical Rayleigh number for this aspect ratio which is approximately 657.5.

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Veronis used a Fourier expansion method and Moore and Weiss used a finite-difference method applied to the stream function and the vorticity. The results of the adapted A.D.I. technique are for a 17 x 17 grid, and those of Moore & Weiss are for a 24 x 18 grid. The values are in good agreement.

The calculations were ended when the Nusselt number was constant from layer to layer to 1 part in 10^8. Typically this corresponded to a time \( t = 6.4 \) in the dimensionless units adopted. For \( R_a \) between 2.0 and 6.0 we used a timestep \( \Delta t = 0.032 \), and for \( R_a \) between 8.0 and 10.0 we used \( \Delta t = 0.016 \). These calculations took from 20 to 30 seconds of CPU time on the CDC 7600 of the University of Manchester Regional Computing Centre. Roughly 75% of this time was spent in solving the pressure equation.

Summary

We have described a numerical scheme which is second-order in both space and time. The use of Fast Fourier Transform techniques for the solution of the pressure equation guarantees accurate incompressibility at all time and enabled us to consider using iteration for part of this scheme. The iterations converge satisfactorily for values of the timestep of the order of one-half to one-quarter of the space step. Numerical calculations are being undertaken to clarify the range of Reynolds numbers and timestep over which the iteration converges.

Acknowledgement is made to Dr. R.A. James for the use of his Poisson solver. I am grateful to Dr. James, Dr. D.L. Moss and Dr. I. Gladwell for their guidance and advice.

References


A formal solution of this equation may be written
\[ I(\mathbf{v}, \mathbf{r}) = \sum_{n=0}^{\infty} I_n(\mathbf{v}, \mathbf{r}) \]

Here, \( I_n(\mathbf{v}, \mathbf{r}) \) is the monochromatic optical path length between the points \( \mathbf{r}' \) and \( \mathbf{r} \), so the super-function itself is a linear functional of the radiation field. The super-function must be a known function of the monochromatic radiation field, \( I(\mathbf{v}, \mathbf{r}) \), so we may write
\[ S_0(\mathbf{v}, \mathbf{r}) = \mathcal{A}(\mathbf{v}, \mathbf{r}) \]

where \( \mathcal{A}(\mathbf{v}, \mathbf{r}) \) is a known function of the monochromatic radiation field, \( I(\mathbf{v}, \mathbf{r}) \), so we may write
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In the present case, the super-function is the monochromatic optical path length between the points \( \mathbf{r}' \) and \( \mathbf{r} \). The super-function must be a known function of the monochromatic radiation field, \( I(\mathbf{v}, \mathbf{r}) \), so we may write
\[ S_0(\mathbf{v}, \mathbf{r}) = \mathcal{A}(\mathbf{v}, \mathbf{r}) \]

Thus, we may write
\[ A = \left[ -\mathcal{A}(\mathbf{v}, \mathbf{r}) \right] \int_0^\infty \mathcal{A}v\mathcal{A}(\mathbf{v}, \mathbf{r}) \, dv + \mathcal{A}(\mathbf{v}, \mathbf{r}) \]

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\[ S_0(\mathbf{v}, \mathbf{r}) = \mathcal{A}(\mathbf{v}, \mathbf{r}) \]

Thus, we may write
\[ A = \left[ -\mathcal{A}(\mathbf{v}, \mathbf{r}) \right] \int_0^\infty \mathcal{A}v\mathcal{A}(\mathbf{v}, \mathbf{r}) \, dv + \mathcal{A}(\mathbf{v}, \mathbf{r}) \]
where the bar denotes the Fourier transform. In writing (9) I assume that depends only on the distance $|r - r'|$ between source and field points; the term $\lambda^* k$ can thus be factored. A factorization of $\bar{S}_{k-1}$ will generally not be possible.

The rate of convergence of the perturbation series may be estimated from the ratio

$$\frac{\bar{S}_{k-1}}{\bar{S}_k} = 1 - \frac{\bar{S}_k}{1 - \lambda^*}$$

(10)

The perturbation series will converge if $|\bar{S}_{k-1}/\bar{S}_k| < 1$ for all $k$; it may converge if this inequality is violated at some wavenumbers, for small $k$. If we assume that $0 < \lambda^* < 1$, criterion (10) may be rearranged to give

$$\bar{S}_{k-1} > \bar{S}_k > (2\bar{S}_k - 1)\bar{S}_{k-1}$$

(11)

If $\lambda^* \approx 0$, the perturbation series is equivalent to the Neumann series solution of the transfer equation. For problems of astrophysical interest, the albedo $\bar{\zeta} < 1$ and the Neumann series converges very slowly. The use of a non-zero $\lambda^*$ may be regarded as a procedure to accelerate convergence of the Neumann series. The RHS of (11) shows that any $\lambda^*$ which satisfies

$$0 < \lambda^* < \frac{1}{2}(\bar{S}_{k-1} + \bar{S}_k) / \bar{S}_k$$

(12)

will lead to a convergent series. The range of possible operators is thus quite large.

### 4. Implementations and Applications of the Model.

The direct solution of Equation (3) is given by $S = (1 - \lambda)^{-1} q$. The prime difficulty of the transfer problem lies in constructing the inverse operator $(1 - \lambda)^{-1}$. The advantages of the perturbation scheme are realised by choosing $\lambda$ so that the inverse operation $(1 - \lambda^*)^{-1}$ is easy to perform. The operator $\lambda$ is applied only in its direct form, and this circumvents the often prohibitive problem of constructing $(1 - \lambda)^{-1}$.

The angle and frequency quadrature perturbation methods described by Cannon (1973a, 1973b) are examples of applications of the general perturbation method. Cannon's techniques use very low order discrete quadratures over $v$ and $\bar{q}$ in the $\lambda^*$ operation. This very significantly reduces the time needed to calculate $(1 - \lambda^*)^{-1} k$. Solutions of very high accuracy may be obtained by using high order quadratures in $\lambda$, and the total time involved is only slightly longer than that required to generate the low order solutions. Cannon's implementation in terms of Feautrier's equation has the advantage that the operation $(1 - \lambda^*)^{-1}$ may be stored.

The redistribution perturbation technique described by Cannon et al (1975) uses complete redistribution in the $\lambda^*$ operation and arbitrary redistribution functions in $q$. This procedure allows problems with any redistribution function to be solved by methods developed to solve the complete redistribution problem.

The perturbation technique described by Cram and Lopert (1976) uses the $\lambda^*$ operator for a constant-property, stationary atmosphere to solve problems wherein the parameters of the transfer equation (velocity, source and sink terms, opacity, etc.) vary with depth. This form is particularly useful in solving hydrodynamic problems, since the effects of time-varying parameters are included only in the direct $\lambda$ operation.

The author is currently extending the perturbation method to treat multi-dimensional transfer problems and problems involving polarized light. In the first extension, a suitable $\lambda^*$ operator may be derived from the kernel approximation to the multi-dimensional Green's function of the problem: a simple-minded application of the one-dimensional operator does not converge. In the extension to treat the non-LTE transfer problem for polarized light, it seems probable that the $\lambda^*$ operator need only be applied to the total intensity component of the Stokes' vector, since this component dominates the transfer problem.

Acknowledgements.

I was introduced to the perturbation method by C.J. Cannon. Applications of the method have resulted from collaboration with him, and with I.M. Vardavas and P.B. Lopert. I am deeply grateful for the support provided by a CSIRO Postdoctoral Studentship and by Sacramento Peak Observatory.

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Two Dimensional Modeling of Tokamaks

by

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The code GM2 is used to calculate the diffusion stages of Tokamak discharges as described by Grad & Hogan [1]. The Pfirsch-Schlüter transport coefficients given by Hazeltine & Hinton [2] are used in a calculation of separatrix formation.

Represent the magnetic field in the conventional form

\[ B = fV\psi + V\psi \times V\psi \]

and let the potential \( \psi \) be determined by the pressure balance equation \( \nabla P = j \times B \) which is rewritten

\[ \frac{V\psi}{r^2} + 4n \psi + \frac{f' r}{r^2} = 0 \quad (\text{Grad-Shafranov}) \]

As is well known, specification of \( P(\psi) \), \( f(\psi) \) and a boundary condition then determines \( \psi \). We chose to specify the location of the outermost flux surface and then determine the value of \( \psi \) there by requiring the total current contained in the problem to be a specified function of time. As in Lagrangian hydrodynamics the position of the grid is the solution to the problem.

In this case the grid consists of selected contours of \( \psi \), intersected by an arbitrarily chosen orthogonal set of curves. GM2 uses Potter & Tuttle's [3] orthogonalization routine to provide these orthogonal lines. The equilibrium equation is linearized and solved by the ADI technique on an orthogonal grid. The grid is then moved to lie along the calculated \( \psi \) contours and the procedure is iterated to take account of nonlinearities.

We see that the fields and currents are thus known as functions of time if \( P(\psi, t) \) and \( f(\psi, t) \) are known. These are determined by the conservation laws once the transport is specified. We need to follow these functions in time on a flux surface, so we integrate the conservation laws over a differential volume surrounding a moving surface to find

\[ \frac{D(nv')}{Dt} + \frac{3}{\psi} (v' \nabla \cdot \psi) = 0 \quad \text{(mass conservation)} \]

\[ \frac{D(P_e v')}{Dt} + P_e \frac{Dv'}{Dt} + \frac{3}{\psi} \left[ v' \left( \nabla \cdot \psi \right) + \frac{5}{2} T_e \frac{\nabla}{\nabla_c} \nabla \cdot \psi \right] = (H - Q) v' \]

\[ \frac{D(P_i v')}{Dt} + P_i \frac{Dv'}{Dt} + \frac{3}{\psi} \left[ v' \left( \nabla \cdot \psi \right) + \frac{5}{2} T_i \frac{\nabla}{\nabla_i} \nabla \cdot \psi \right] = Q v' \]

\[ \frac{Dfv'}{Dt} + \frac{3}{\psi} (v' \nabla \cdot E) = 0 \]

In the above, \( v(\psi) \) is the volume contained inside flux contour \( \psi \) and \( v' \) is the derivative of this function. \( \frac{D}{Dt} \) is the time derivative on the moving grid, i.e., at constant \( \psi \). \( T_c \) is the collisional part of the particle flux, \( Q_e \) and \( Q_i \) are heat fluxes, \( H \) is Joule heating, \( Q_\Delta \) is ion-electron heat exchange, and \( E \) is the electric field. These are all related to gradients of \( n, P_e, P_i \) and \( f \) by the Hazeltine-Hinton formulae, closing the system.

These equations hold in the presence of a Doublet separatrix, provided \( v(\psi) \) is interpreted to mean the volume contained within both lobes, and provided up-down symmetry exists so the lobes are equivalent. Unfortunately, the orthogonalizer has not yet been taught how to handle the separatrix, so we cannot run such problems on the code. We expect to be able to do so in the near future, but until then we can study the formation stages of the separatrix, realizing that the calculation must stop when one forms.

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Let us consider a Doublet III class machine, major radius 140 cm, minor radius 45. Assume a field of 3.2 T, a central density of $2_{14}$, initially uniform 50 eV temperature, effective $\varepsilon$ of 4, and 100 kA of current, rising linearly to 170 kA at 10 ms. Assume the current is initially distributed proportional to $r$.

Let there be a 13 cm dimple in both sides of the plasma.

We find the initial equilibrium flux surface shapes go from the hourglass form imposed at the outer boundary to kidney shapes near the magnetic axis. This axis is shifted to the outside of the plasma by the vertical field implied for equilibrium.

As the problem progresses the inner flux surfaces begin to pinch in at the waist, until at about 7 ms the calculation is stopped by the zone crossing on axis. This is about one energy confinement time at these temperatures. The 150 kA now in the plasma has heated the center to 70 eV, and has diffused to a rather more uniform distribution. Final contours and current distributions are shown in Figure 1.

References:


1. Introduction

Reversal of the impurity flux as a possible mean to avoid large impurity concentrations in a Tokamak plasma has been discussed already in several papers \[1, 2, 3\]. Flux reversal can in principle be obtained a) by additional heating, either R.F. heating or neutral injection, leading to a steepening of the ion-temperature profile and thus to an enhancement of the thermal diffusion b) by influx of neutral gas from the outside.

In the following the combined effects of additional heating by neutral injection and of neutral gas influx on the impurity transport in a tokamak plasma (TEXTOR) is considered as well as, for comparison, the effect of each measure alone.

2. Neutral injection

2.1 Calculation of the beam deposition

The beam axis lies in a plane \( z = z_0 \) and is tangent to the torus at \( R = R_t \) (fig. 1). The cross section is either circular (radius \( R_0 \)) or rectangular (half height \( h \) and half width \( w \)).

The interaction of the injected hot neutrals with the plasma leads to hot ions which are deposited on the drift surfaces. The interaction is determined by the cross section for electron ionization, proton ionization and charge exchange \( 1/4, 5, 6 \). Following Rome, Callen, Clark and R.R. Smith \( 5, 6 \) the hot ion birth rate is calculated from the above mentioned data, the energy of the injected neutrals on which the cross sections strongly depend, the beam current strength, and the plasma parameters which are consistently calculated by Dühs' code. The main idea in this calculation is that the density \( n_p \) of the beam particles at a distance \( S \) from the beam entrance is \( \frac{S}{n_p} \) in the pencil beam case given by

\[
n_B(S) = n_{B0} \exp \left[ -D(S) \right]
\]

where the absorption decrement \( D(S) \) is given by the integral

\[
D(S) = \int \frac{dS}{\Lambda(S)}
\]

(2.1.2)

The quantity \( \Lambda(S) = 1/(n_0(S) \sigma_n) \) is the mean free path of a injected neutral, \( n_0 \) the particle density and \( \sigma_n \) the total cross section. The "negative derivative" \( d n_p/dS \) is then the hot ion birth rate per unit length.

2.2 Thermalization and charge exchange of the hot ions

Through the interaction of the hot ions with the background plasma, their density decreases by charge exchange, their energy by coulomb collisions. The slowing down is much more faster than the attenuation of the density. The particle energy decreases according to the equation \[1/7\]

\[
de/dt = -\left( K\epsilon^2 + B\Sigma n_T/e \right) S^{3/2}
\]

(2.2.1)

where the quantities \( K, B \) are functions of the plasma parameters, \( T_e \) is the electron temperature. The decay of the density \( n_{p0} \) is given by

\[
d n_p/dt = -Q_\alpha n_{p0} + \sum_j \left( n_j \right)
\]

(2.2.2)

\( Q_\alpha \) is the rate coefficient for charge exchange, the expression \( \sum_j \) the density of the neutral gas including cold neutrals \( n_{p0} \) and four generations of hot neutrals \( n_{p0}/S^4 \).

It is assumed that the "very" hot neutrals penetrate the plasma without further interaction. Thus they can be included in the flux of the recycled cold neutrals.

3. Plasma transport

For the calculation Dühs' \( G \)-regime-code is used \( 8/1 \) which takes into account the drift and the trapped particle instabilities. The transport equations also contain source terms for the particle and energy gains or losses because of ionization, recombination, radiation, charge exchange and beam heating. The particle gain arising from the beam deposition and the charge exchange loss according to (2.2.2) has been added.

4. Impurity transport

The local density \( n_p \) of the impurity ions, summed over all ionization levels, is governed by the diffusion equation

\[
\frac{\partial n_p}{\partial t} = -\frac{\partial}{\partial \tau} \left( \frac{\partial n_p}{\partial \tau} \right)
\]

(4.1)

The impurity flux \( \gamma_p \) according to neoclassical theory is

\[
\gamma_p = \sum_i C_i \frac{\partial n_i}{\partial \eta} \frac{\partial n_p}{\partial \rho} - C_5 \frac{\partial n_p}{\partial \rho} \frac{\partial^2 n_p}{\partial \eta^2} + C_4 \frac{\partial n_p}{\partial \rho} \frac{\partial n_p}{\partial \eta}
\]

(4.2)

The coefficients \( C_i \) are defined as follows /12/:

\[
C_1 = D_{\rho \rho} \quad C_2 = D_{\rho \eta} \langle Z \rangle \\
C_3 = D_{\rho 0} \langle Z^2 \rangle - \langle Z \rangle \langle Z \rangle \\
C_4 = D_{\rho 0} \langle Z \rangle \frac{\partial \langle Z \rangle}{\partial \rho} \\
C_5 = \left[ 4.5 \langle Z^2 \rangle - 4 \langle Z \rangle \right] D_{\rho \rho} + D_{\rho 0} \frac{\partial^2 \langle Z \rangle}{\partial \rho^2} / T_e \\
C_6 = \left[ 2 \langle Z^2 \rangle - 2 \langle Z \rangle \right] \left( 4.5 D_{\rho 0} - D_{\rho \rho} \right) / T_e
\]

(4.3)

The quantities \( D_{\rho \rho}, D_{\rho \rho}, D_{\rho 0}, D_{\rho 0} \) describe the interactions of the plasma with the impurities and among the impurities /12, 9/. The averages \( \langle Z \rangle = \langle \Sigma Z \rangle n_z / n_T \) and \( \langle Z^2 \rangle = \langle \Sigma Z^2 \rangle n_z / n_T \) are calculated from the rate equations /10/. \( Z_j \) and \( n_{p0} \) are
the charge number and the density of one ion species. If the impurities are completely ionized, the average impurity (oxygen) density is about 2% of the average plasma density and the plasma edge is dominated by the impurities. The impurities are in the banana-regime where we have roughly $\frac{C_0}{C} = 0.4/\tau_I$. Hence the impurities can diffuse outward if

$$\frac{C_0}{C_0} > \frac{2}{\tau_I} \frac{2 \partial n_e}{\partial x} - \frac{2}{\partial \tau_I} \frac{2 \partial n_v}{\partial x}$$

(4.3)

where $C_0$ is the charge number of the completely ionized impurity. When the impurities are in the plateau regime, the right hand side of (4.3) is multiplied by a factor of order unity. When the plasma is in the plateau regime, the coefficient $C_0$ changes sign $/2$; the thermal diffusion is then directed inward, this is always the case in the plasma center.

5. Results

Calculations were done for several sets of data characterizing the impurities, the neutral gas influx and the neutral injection. Some typical results are given below.

For the computations TEXTOR data have been taken which have been a = 50 cm, $R = 175$ cm, $I = 476$ kA, $B_0 = 20$ kG. As initial values parabolic profiles for densities, temperatures, and the current are used. The total toroidal current is assumed to be constant in time, (constant current model). The boundary conditions for the impurities $\Gamma_{0}^{(w)} = 0$. The average impurity (oxygen) density is always 2% of the average proton density at $t = 0$.

In Fig. 2 and Fig. 3 the impurity profile $n_{v}$ and the ion density $n_{i}$ and the temperature profiles $T_{e}$ and $T_{i}$ are shown at $t = 400$ ms.

a) By application of neutral gas influx from 100 to 400 ms, the average proton density is raised from $1.5 \times 10^{19}$/cm$^3$ to $3 \times 10^{19}$/cm$^3$ (Fig. 2).

b) In addition to the neutral gas influx between 200 ms and 400 ms a 'uniform (circular)' beam with $I_{b} = 48$ A, $E_{b} = 50$ keV, $n_{b} = 25$ cm, $R_{b} = 175$ cm and $r_{b} = 25$ cm has been switched on (Fig. 3).

We see that application of the beam in addition to the neutral gas influx shifts the peak of the impurity density appreciably towards the plasma edge; the impurity density at the minor axis decreases by a factor 0.5. In Fig. 3 the fluxes proportional to the impurity gradient ($F_{i}$), $\Gamma_{0}^{(w)}$, the ion density ($F_{i}$), to the ion temperature gradient ($F_{e}$) and the total flux ($F_{w} + F_{x}$) are shown. Evidently $F_{i}$ is positive and greater than $F_{w} + F_{x}$ in the outer region of the plasma. In all figures the normalizing factors are given below.

If the beam is applied alone, the impurity profile becomes at the plasma edge approximately as steep as in Fig. 3, but no impurity density minimum occurs at the plasma center. The flux $F_{i}$ is approx. the same as shown in Fig. 3. Using uniform a beam with rectangular cross section ($n_{b} = 12.5$ cm, $w = 5$ cm, $r_{b} = 25$ cm) hot ions are born at the plasma center. This leads to a shift of the maximum of the ion temperature (in Fig. 3, located at $r = 25$ cm) towards the plasma center. The steepening of the temperature profile which is not the case in the circular beam is not sufficient to invert the impurity flux in the vicinity of the plasma edge.

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Fig. 1 Geometry of injection

Fig. 2 Plasma Parameters without beam;
$N_{i} = 761$ eV, $N_{e} = 857$ eV,
$N_{i} = 3.57 \times 10^{17}$/cm$^3$, $N_{e} = 4.53 \times 10^{11}$/cm$^3$

Fig. 3 Plasma parameters with beam;
$N_{i} = 1.95$ keV, $N_{e} = 1.75$ keV,
$N_{i} = 4.27 \times 10^{15}$/cm$^3$, $N_{e} = 4.26 \times 10^{11}$/cm$^3$

Impurity fluxes with beam;
$F_{i} = 4.85 \times 10^{12}$/cm$^2$/sec
FINITE ELEMENT APPROACH TO THE QUASI-LINEAR EVOLUTION OF THE
GENTLE BUMP INSTABILITY IN A 2-D PLASMA

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Although the basic equations of the quasi-linear theory of plasma turbulence were formulated more than one and half decade ago [1,2], it is not yet much known about their solutions for non-one-dimensional systems. By analytical means, only certain general features of possible asymptotic states of the system can be deduced [3,4].

The relaxation dynamics as well as the final state of the system remain unknown. In the present work an attempt is made to attack these problems by a numerical procedure. In particular, the evolution of the gentle bump instability in a non-magnetized 2-D plasma is investigated within the framework of the quasi-linear equations by means of the finite element method.

On making use of the following scaling: $k = \kappa/\sqrt{\alpha}$, $\nu = \nu_{th}^{'}$, $\tau = \tau/\nu$, $f = f/\nu_{th}^{'}$, $I = I/4nT$, the fundamental equations assume the form [5]

\begin{align}
(1) \quad & \frac{d\kappa}{d\nu} + \frac{d\nu}{d\tau} + D \frac{d\nu}{d\tau} = \frac{\kappa}{k} \sum_k \kappa_k \sum_{\nu_{th}^{'}(k)} \delta(\nu - \nu_{th}^{'}(k)) \\
(2) \quad & \frac{d\kappa}{d\tau} - 2 \nu_{th}^{'}(k) \frac{d\kappa}{d\tau} = \frac{1}{k} \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k) \frac{d\nu}{d\tau} \delta(\nu - \nu_{th}^{'}(k)) \frac{d\nu}{d\tau}.
\end{align}

Here all symbols have their standard meaning. The equations (1) and (2) have three integrals of motion: the conservation of particles, momentum, and energy. We write only the latter here as $\frac{dW}{dt} = \frac{dW}{dt} = \frac{dW}{dt} = \text{const}$, where $W = \frac{1}{2} \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k) \kappa_k \kappa_k \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k) \kappa_k \kappa_k$. Moreover, for diagnostic purposes we also define the quantity $<k> = \frac{1}{W} \sum_{\nu_{th}^{'}(k)} \kappa_k \kappa_k \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k)$.

Next, we put the equation (1) into the Galerkin (weak) form [6]: At each time $t > 0$,

\begin{align}
(3) \quad & \int_{\Omega} \frac{d\nu}{d\tau} \left( \frac{d\kappa}{d\tau} + \frac{d\nu}{d\tau} + D \frac{d\nu}{d\tau} \right) = \text{boundary term}
\end{align}

for all $\kappa$ in $H^1$, where $\Omega$ is a finite domain in the velocity space. In what follows we neglect the boundary term since it can be made small by taking sufficiently great $\Omega$. In expanding the unknown solution as $\kappa = \sum_{\nu_{th}^{'}(k)} \kappa_k \kappa_k \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k) \kappa_k \kappa_k$, where $\kappa_k$ are linear (pyramid) basis functions of an $N$-dimensional subspace of $H^1$, and choosing $g = \kappa_k$ ($1 \leq k \leq N$) we transform the equation (3) into a system of ordinary differential equations in time. It reads

\begin{align}
(4) \quad & \frac{d\kappa}{d\tau} = \frac{\kappa}{k} \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k) \kappa_k \kappa_k \sum_{\nu_{th}^{'}(k)} \nu_{th}^{'}(k) \kappa_k \kappa_k
\end{align}

We point out that for convenience the meshes attributed to the basis functions in the domain $\Omega$ are chosen to be non-equidistant.

The equations (2) and (4) were solved by an implicit, finite-level, time-centred, finite difference scheme. The initial distribution function was chosen as

\begin{align}
(5) \quad & f(t=0) = \frac{1}{2} \sum_{\nu_{th}^{'}(k)} \exp(-\nu_{th}^{'}(k)/2) + \xi \exp(-\xi \left[ \left( \nu_{th}^{'}(k) - \nu_{th}^{'}(k) \right) \nu_{th}^{'}(k) \right])
\end{align}

the initial spectral function as $I(0) = \text{const}$.

The values of the parameters $\xi$ and $\nu$ were 0.15 and 4.5, respectively. The number of basis functions $N$ as well as the number of wave modes $M$ were varied in certain ranges, typically between 300 and 600. The initial value of the wave energy $W(t=0)$ was taken to be $3 \times 10^{-3}$.

The evolution of the system was followed in all the cases during the time $t \leq 600$. The accuracy of the solutions was verified by checking the variations of the integrals of motion and repeating a number of computations with different values of $N$ and $M$. The behaviour of the wave energy is shown in Fig. 1.

Here the lines a and c represent the cases with $N=300$, $M=300$ and $N=600$, $M=600$, respectively. On comparing these two results we can conclude on a good convergence of the numerical procedure with respect to $N$ and $M$. In order to investigate an influence of the boundary of the $\Omega$ domain upon the evolution of the system we modified the basis functions in such a manner that they were equal to zero on the boundary. This set of computations ($N=300$, $M=300$) is represented by the line b. We see that towards the end of the time considered the boundary effects begin to play a role.

Figure 2 displays the quantity $<k>$, which characterizes the location of the wave energy spectrum in $k$-space, for the same cases as in Fig. 1. We notice that all the time the spectrum moves up several half-widths along the $k$-axis. Simultaneously, the widths of the spectrum in both directions (not shown in the figure) narrow by a factor 2 approximately. To our knowledge these two phenomena were never observed in a 1-D system. The variation of the energy integral is plotted in Fig. 3 for the above-mentioned cases again. It indicates that the best accuracy of computations is achieved for great values of $N$ and $M$, and for no modification of the basis functions on the boundary.

The variation of momentum integral exhibits a similar character, whereas the conservation of particles is nearly perfect (12 digits) for the cases a and c.

In a set of computations we have also used finite hybrid elements [7]. However, the
conservation of all three integrals was worse than when using the regular elements.

Generally, we can roughly distinguish two different stages in the evolution of the system. In the first one (0 < t < 30) the wave energy grows monotonically up to the value $W_w \sim 10 W_0 (t=0)$. At the same time a 1-D quasi-plateau is formed on the particle distribution function along the x-axis. During the second stage the value of the wave energy remains approximately constant while the particle distribution function undergoes the diffusion process along the y-axis in a domain between the bulk of particles and the location of the beam. This phenomenon is displayed in Fig. 4.

Since we had to deal with a lot of damped waves in all computations we imposed a lower bound ("thermal level") on the value of the spectral function $I_k$. The evolution of the system turned out to be sensitive to that value. This fact apparently indicates that for obtaining an unique solution which has a physical meaning one should include terms accounting for the spontaneous emission into the quasi-linear equations.

REFERENCES

The Resistive Tearing Mode by a Two-Dimensional
Finite Difference Method.
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Tearing instabilities of neutral sheets are discussed as
possible mechanisms to explain different eruptive phenomena
such as solar flares or magnetospheric substorms. Detailed
discussion of the stability of the collisionless plasma sheet
in the earth's magnetotail show that two-dimensional effects are
important in stabilizing the plasma sheet (Birn et al., 1975).
The onset of the instability can be explained by the magnetic
field component normal to the neutral sheet becoming smaller
than some critical value. The similarity between the tearing
mode of collisionless plasma sheets and that of resistive plasma
sheets indicates that a similar criterion might hold for the onset
of a resistive tearing mode. Since an appropriate two-dimensional
theory of resistive instabilities does not yet seem to exist,
numerical experiments are desirable to get some information about
this problem. Another reason for numerical experiments is to study
the nonlinear evolution of the tearing instability. The theory of
Collisionless plasmas indicates that a nonlinear effect may be the
conversion of energy from small scale turbulence into larger
scales (Biskamp and Schindler, 1971), and it is one purpose of our
investigations to test this effect by means of numerical studies.

A two-dimensional explicit MHD-code has been constructed
which shall be used to study the above mentioned questions.
The MHD-equations including resistive terms with a small
resistivity \( \eta \) are used in the following form:

\[
\frac{\partial p}{\partial t} = - \nabla \cdot (\rho \mathbf{v})
\]

\[
\frac{\partial \mathbf{v}}{\partial t} = - \nabla (p + \frac{1}{\mu_0} \mathbf{B}^2) - \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \frac{1}{\mu_0} \mathbf{B} \mathbf{B})
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{4}{\mu_0} \nabla T
\]

\[
\frac{\partial \mathbf{h}}{\partial t} = - \nabla \cdot (\mathbf{h} \mathbf{v}) + \frac{1}{\gamma} \frac{1}{\mu_0} \mathbf{h} \left( \frac{1}{\mu_0} \mathbf{B} \mathbf{B} \right)^{\gamma-1} \left( \mathbf{v} \times \mathbf{B} \right)^2 \quad \text{with } h = p^{1/\gamma}
\]

All but equation (4) are written in conservative form, so that
the fluxes are automatically conserved. The energy equation (4)
is written in such a form that the only non-conservative term
contains the small factor \( \eta \), so that the deviations from energy
conservation should be small. Instead of (4) an isothermal
equation of state \( p-c \) is often used.

The symmetry plane of the plasma sheet is used as one boundary
of the finite system. The opposite boundary is set to be a solid
wall. In the first step periodicity conditions are used for the
remaining two boundaries. These shall be replaced by more general
conditions at a later stage.

There are several difficulties to be overcome:

The problem contains two different time scales which have to
be resolved. The typical time scale of the tearing mode lies
between the time scales of MHD-waves and of magnetic diffusion
which should be well separated to identify the tearing time
scale. This requires a magnetic Reynolds number \( S \) of about 1000
or larger. Since we are interested in the stability behaviour
on the tearing time scale it seems however that high numerical
accuracy in the description of MHD-waves is not required pro-
vided that the time step is small enough not to introduce nu-
merical instabilities which cover the physical instabilities.

The problem contains also two different length scales.
Essential for the existence of the tearing instability is a
thin layer around the neutral sheet in which the ideal MHD
condition \( \mathbf{B} \cdot \mathbf{v} = \mathbf{0} \) is violated. The thickness of this layer
compared to the thickness of the plasma sheet \( 2L \) is of the order
of the ratio of \( L \) to the wavelength \( \lambda \) of the tearing mode.
Theory shows that the tearing mode will grow unstable only if
this ratio is small enough compared to unity. So there should
be enough meshpoints to resolve this thin layer. On the other
hand the envisaged applications require that the distance to the
boundary should be large compared to the thickness of the plasma
sheet. This difficulty can be overcome with a reasonable number
of meshpoints only if a non-uniform mesh is used. This has been
obtained by means of a variable transformation $y + z = f(y)$ and a
substitution of the corresponding difference quotients $\frac{\Delta u}{\Delta y}, \frac{df}{dy}, \frac{\Delta u}{\Delta z}$. By this means the boundary can even be moved to infinity.

The code should provide minimal damping of physically unstable modes so that numerical diffusion does not inhibit the growth of these instabilities. This was one reason for choosing a leapfrog scheme.

Unfortunately this leads to the disadvantage that there is no damping of non-physical oscillations between neighbouring grid points which appear largest in the plasma velocity and may grow unstable. So some effort is needed to avoid these oscillations without smoothing of the larger scale physical variations. The best results were obtained by adding a viscosity term $\nu \frac{\partial^2 V}{\partial y}$ to equation (2), which is non-zero only if oscillations between neighbouring grid points occur. This can be done in a flux conserving form if the viscosity term is constructed from the fluxes $\nu V$. The flux in one direction is compared with the fluxes at neighbouring grid points in the same direction and kept non-zero only if its sign changes at both sides.

As a first test of the code the growth of the tearing instability from an arbitrary perturbation of a one-dimensional unperturbed state has been studied. Since the unperturbed magnetic field diffuses, the characteristic thickness $2L$ of the plasma sheet grows with time. Theory (Furth et al., 1963) predicts that in the region of maximum growth rate the growth time $\tau$ should be proportional to $L^{3/2}$. Figure 1 shows the dependence of $\tau L^{-3/2}$ on time $t$ for two examples with different Reynolds number $S$. An almost constant value is reached after the eigenmode has grown larger than the initial disturbance.

Figure 2 shows experimentally obtained growth rates $\alpha$ for different wavelengths $\lambda$ (resp. $\alpha = 2\pi L/\lambda$) which are in good agreement with theoretical results of Furth et al. (1963). The variation of the experimentally obtained growth rates with time has been eliminated by extrapolation of the function $\tau(L)$ to the initial value $L$. The results confirm that the diffusion does not strongly influence the evolution of the tearing mode (Barston, 1972).

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LARGE SCALE SHOCK WAVES IN BARRED GALAXIES

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Considerable progress has been made in the field of galactic dynamics during recent years, where especially the formation and persistence of spiral structure have been explained in terms of density wave theory. Most of these works have been concentrated on the structure of normal spirals while less attention has been paid to barred spirals. In the case of normal spirals one can assume that the pitch angle of spirals is so small that the partial differential equations governing the dynamics reduce to ordinary differential equations. This assumption, however, does not hold in the case of barred spirals. Therefore one must solve at least two dimensional (in space) partial differential equations.

The currently accepted idea for the theory of barred spirals is as follows. The mechanism responsible for the formation of spiral arms in barred galaxies is completely different from that of ordinary spiral galaxies. If a bar is rotating sufficiently rapidly, then gas may be ejected through the end of the bar and the ejected gas will form spirals. The spirals formed by this outflow is nothing but material arms.

There are several evidences which make such an outflow theory rather improbable. There exist younger stars in the outer part of the spiral arms, which are too young to come from the bar. These stars must have been formed in the arms. Secondly, dark lanes in the bar are sometimes observed. To associate these dark lanes with shock waves is very natural, but such shocks are difficult to be explained in terms of the outflow model. Furthermore, there exist dark lanes in the trailing side of spiral arms and HII regions in the leading side. These are also explained in terms of shocks but difficult to understand in outflow models.

Considering these facts we are led to the idea that the spirals in barred galaxies are also density waves as well as ordinary spirals. But the difference is that those spirals are excited mechanically by the bar and the self gravity of spiral itself may not be important. In order to test this idea, we followed gaseous motions in the given potential field made by a stellar bar (plus a stellar disk). The self gravity of gas is neglected. The gas is assumed to be isothermal. We assume that the thickness of the gas layer is constant, therefore the equations reduce to two-dimensional hydrodynamic ones. The bar has a prolate spheroidal figure with heterogeneous density; the potential is given in analytic form. Let us introduce four non-dimensional quantities following de Vaucouleurs and Freeman

\[ p = \frac{GM_{\text{disk}}}{(\omega c)^2 K}, \quad q = \frac{105 GM_{\text{bar}}}{32(\omega c)^2 \epsilon}, \quad a/c, K/\epsilon \]

where \( \epsilon = (a^2 - c^2)^{\frac{1}{4}} \), \( a \) and \( c \) are semi-major and semi-minor axis of the bar, \( \omega \) the rotational speed of the bar, \( K \) the length scale of the disk. The parameter \( p \) and \( q \) represent the ratio of the gravitational energy to the rotational energy.

The method of computation is Fluid-in-Cell method developed by Los Alamos group. This method is very stable and consumes little computer time but may produce rather large artificial diffusion. The number of mesh is about 50 x 30. The initial density distribution is uniform. The initial velocity field is so chosen as close as possible to the final velocity field.
After a few rotations of a galaxy, the system settle into quasi-steady state. Two cases are shown as examples: case (1) \( P = Q = 10, \ K = e, \ a/c = 3 \), case (2) \( P = 20, Q = 4, K/e = 1/3, a/c = 2 \). The arrows are velocity vectors, contour lines are iso-density contours, \( X \) are co-rotation points. At hatched regions, the value of artificial viscosity is very big, so shocks are expected to occur.

From these computations we can conclude that: 1, the dark lanes observed in the bar is naturally associated with the shock 2, the spiral arms are density enhancement just behind the shocks, 3, these shocks are induced by non-axisymmetric gravitational field due to the bar, therefore the self gravity is not important for the formation of spiral arms, 4 in some cases, we obtain four spiral arms.

Another interesting aspect of the present calculation is a loss of angular momentum of gas at the shock. The gas rushing into shocks lose energy by radiating it out and spiral into the centre. In this way gas loses angular momentum and give it to stellar component. There is an observational evidence that at the centre of a barred galaxy there is plenty of gas while this is not the case in ordinary spirals.

Reference

2 - LAGRANGIAN SIMULATION

In order to be able to describe the non-linear zone, we use a Lagrangian simulation. To solve this problem, we used a method to determine the bounded region. One has to assure that this bounded region gives a representative result. The assumed zone is shown in figure 2.

YZ being a plane of symmetry, we restrict the region to $x > 0$. Hence it follows that the osculation of the plane $x = 0$ is obtained. The region is bounded by the lines $y = 0$, $z = 0$, and $z = x$. The region is bounded by the lines $y = 0$, $z = 0$, and $z = x$. The region is bounded by the lines $y = 0$, $z = 0$, and $z = x$. The region is bounded by the lines $y = 0$, $z = 0$, and $z = x$.

As for the BC and AB sides, they must be taken for the unperturbed plasma region, and according to the assumed boundary conditions, the potential is $0$ everywhere. In fact, the plasma zone would be too large, so we took the potential $0(x, z)$, as given by the linear theory, on the zone border.

As already pointed out, the simulation zone must be sufficiently large in order that the imposed boundary conditions do not disturb the actual phenomena. Another constraint is that the network members have to be small enough to make the characteristic physical lengths. To escape computer time, we chose the following compromise:

- $\Omega = 0.75$, $\nu = 45/14$, $\beta = 2$.
- $A_B = 24 A_B$.
- The unperturbed ions are replaced by a continuous positive background, while the electrons are taken as finite-size particles having the same dimensions as the meshes according to the CIC method.

At the beginning, the electrons are injected on the AB side with a random velocity distributed according to a reduced Maxwellian about the mean value $\langle V_x \rangle = 0, \langle V_x^2 \rangle = 2 Z$. For $Z < 0, V_x$ a region, the dynamic sheath, where the electric potential $\phi$ is independent of $Z$ and varies with $X$ according to the same law as in a plasma at rest. If they leave the zone through the AB side, the electrons are lost. If they leave the zone through the BC side, they are reflected or lost according to whether the charge density locally positive or negative.

The simulation is stopped when the phenomena become steady.
3 - RESULTS

In the precursor region, upstream from the obstacle, one can see on figures 3 and 4 that even for strongly negative obstacle potentials, the points of the simulation remain approximately on the solid-line curve corresponding to the linear theory.

In the static sheath region, outside the linear field, \( \phi \) and \( X \) are related by

\[
X = \int_0^\phi 2^{-1/2} \left[ \exp(u) - u - 1 \right]^{1/2} \, du
\]

or

\[
\phi = \Phi_\infty (\phi_0 | X)
\]

which defines \( (\phi_0, X) \mapsto \Phi_\infty \)

Figure 5 shows that the points of simulation are located on the curves deduced from (2) at least as long as \( X \) is not too large. The dotted-line is the plot of (1).

Lastly, in the region of the dynamical sheath, one finds a potential given by:

\[
\phi = \text{Inf}(\Phi_\infty (\phi_0 | V, X, Z), \Phi_\infty (\phi_0 | X))
\]

This result is clearly pointed out on figure 6 where the solid curve represents (1) and the dotted-line curves (2).

In conclusion, the potential is given by (3) in the half-space \( Z < 0 \) (downstream from the leading edge, the so-called dynamical and static sheath regions successively) and by (1) for \( Z > 0 \) (upstream form the leading edge: precursor region). In that precursor region, (1) is valid even for \( |\phi_0| \ll kT/e \).
Traces for Radiative Processes in Pulsar Magnetosphere
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Abstract
Radiative processes in the magnetosphere of pulsars require a special symbol manipulation technique to evaluate the traces, which differ from the common case for which procedures are available.

In studying pulsars and especially possible radiative processes, one realizes that in actuality the processes are governed by high energy particles, as particles can easily be accelerated to relativistic velocities due to the large distances traversed by them. In relativistic calculations that involve radiative processes, the electrons play a major part whether it is an emission process due to Coulomb interactions with the heavy ions or mode conversion where longitudinal waves generated by instability in the plasma are converted into transverse photons [1]. The instability in the plasma itself can be traced to the gas being penetrated by a thin electron beam with the density of the order of one percent of the ambient plasma. Needless to say that in any such process one has to sum over the spin states in order to compensate for our ignorance of the exact spin states.

The literature provides us with analytic expressions for such traces only to lower numbers of Dirac matrices [2] with the maximal amount of nine such matrices when \( \gamma^5 \) is written explicitly, namely, \( \text{Tr} \gamma^5 \gamma^5 \ldots \gamma^5 [3] \), and it is obvious that automated computation is utilized in realistic calculations.

In fact tracing over Dirac matrices to any degree allowed by present machine capabilities has become a straightforward operation with the now standard procedures such as ASHMEADIL [4], SCHOONSCHIP [5], or REDUCE [6]. For example Clark and collaborators have used extensively and successfully REDUCE whenever such traces had to be reckoned with [7].

The above standard procedures however cannot be used as they stand, for elementary processes such as the ones described before, where relativistic plasma or strong magnetic fields—where \( \omega_e \gg \omega_p \) are present, where \( \omega_e = \frac{\mathcal{E}}{mc} \) and \( \omega_p = \frac{\sqrt{\mathcal{E} m^2}}{m} \). The reason for the difficulty is, that the square of such electron wave function is irreducible and cannot be diagonalized.

An alternative method would be to expand such an irreducible matrix in the sixteen independent \( \Gamma \) matrices that form a complete set, and then employ the coefficients in the automated computation. Indeed, in general one has in such processes to perform a trace of the form \( \text{Tr} \gamma^j \gamma^k \gamma^l \gamma^m \), where \( \gamma \) is obtained from the solution of the Dirac equation in a magnetic field [8], which is actually \( \text{Tr} \gamma^j \gamma^k \gamma^l \gamma^m \) where

\[
\mathcal{U}_\Gamma \mathcal{U} = \left[
\begin{array}{cccc}
\mathcal{C}^2 \mathcal{H}_n^2 & 0 & \mathcal{C}_4 \mathcal{H}_n^2 & 0 \\
0 & \mathcal{C}_2 \mathcal{H}_n^2 & 0 & \mathcal{C}_4 \mathcal{H}_n^2 \\
\mathcal{C}_3 \mathcal{H}_n^2 & 0 & \mathcal{C}_2 \mathcal{H}_n^2 & 0 \\
0 & \mathcal{C}_3 \mathcal{H}_n^2 & 0 & \mathcal{C}_4 \mathcal{H}_n^2 \\
\end{array}\right]
\]
An are the Hermite polynomials normalized to 1 in \((-\infty, \infty)\) and the \(c_i\) are coefficients involving information about the energy and Landau quantum numbers, \(A\) and \(B\) symbolize the different Lorentz four vectors. For example, in the longitudinal to transverse mode conversion [1] we find that \(A = \psi \phi'\) and \(B = \psi' \phi\) where \(\psi\) and \(\psi'\) are the appropriate polarization vectors of the plasma modes and \(k\) comes from the contribution of the propagator to the momentum, and \(\phi = \int e^k\). Obviously one cannot use here the standard procedure.

For any particular combination of \(A\) and \(B\) one has to calculate 13 terms with the same general coefficients obtained in the expansion of the matrix product \(u \bar{u}\) [9].

In using this method it is possible to disregard specific characteristics of the calculation otherwise needed, such as the conditions imposed by the delta functions, governing transfer among Landau levels, in the consideration of mode conversion for example, without which one cannot proceed with the calculation. Thus, the method though quite straightforward is general and applicable in any elementary process in which the matrix presents such non diagonal terms.

References

NUMERICAL SIMULATION OF RESISTIVE INSTABILITIES OF CHARGED PARTICLE BEAMS
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Introduction

It has been known for some time that a circulating beam of particles can become unstable. The basic mechanism underlying such an instability was first investigated by Maxwell in his famous essay on the stability of Saturn's rings. He showed that whenever the frequency of rotation of particles in a ring interacting via repulsive forces is a decreasing function of their energy, then the ring is subject to an instability. Another situation which can give rise to such a frequency-energy relation is, for example, a relativistic beam of charged particles circulating in proton storage rings or electron ring accelerators. There the interaction of a beam with its surroundings influences the static self-field of the beam and consequently the behaviour of individual particles, whose motion is determined by the combination of externally applied fields and self-consistent beam fields.

Considerable effort has been undertaken to investigate the equilibrium properties of beams in plasmas and accelerator columns. The complexity of the problem suggests the use of numerical techniques and a high speed computer to allow non-linear effects to be studied. Our computational approach to study resistive instabilities consists of following relativistic particles around a strong focusing accelerator, taking into account energy changes resulting from the electromagnetic interaction with an initially unexcited cylindrical cavity and an externally applied electric rf field. In contrast to other computations, our method allows the study of cases where the coupling impedances depend strongly on frequency within the frequency range covered by the beam.

Simulation Program

It is possible to give an adequate description of a beam by employing a number N of superparticles of charge Ze, where e is the elementary charge and Z is chosen so that the quantity

\[ z_0 = \frac{2ZZe0}{\gamma} \]

corresponds to a realistic current circulating in an accelerator with angular frequency \( \Omega \). As illustrated by Fig. 1, the superparticles pass through a cylindrical cavity and subsequently through a rf accelerating gap. Essentially three effects arise from the passage of many particles: firstly, the particles excite electromagnetic fields in the resonant cavity and consequently lose energy and, secondly, each of the particles is accelerated or decelerated by the fields which are already in the cavity due to the passages of earlier particles and of earlier passages of the same particle. Thirdly, the particles suffer from an energy change due to the rf field in the accelerating gap. Either effect changes the energy of the particles and therefore their subsequent revolution frequency. We assume that outside the cavity and the accelerating gap the particles do not interact with their surroundings and that the effect of variations of the axial motion on the transverse motion is negligible.

The distribution of the i=1,2,...,N particles is described by pairs of co-ordinates \( (\xi, E) \), \( \xi = 2\pi t/T \), where \( t \) is the next time the i-th particle of energy \( E \) arrives at the cavity. The energy scale is chosen so that \( E=1 \) for particles in resonance with the cavity, i.e. with revolution time

\[ T = \frac{2\pi}{\Omega} = \frac{2\pi n}{\omega} \quad \text{where} \quad \Omega = \frac{2}{\sqrt{1 - \beta^2}} \]

\( \beta \) is the mean radius of the accelerator, \( n \) is the mode number of the density modulation and \( \gamma \) is the total energy of the particle is units of its rest energy \( m_0c^2 \). The interaction with the resonant cavity and the rf gap modify, for each particle the pair \( (\xi, E) \). The result is a dependence on the vector potential, \( \xi \) and \( \dot{\xi} \). The subscript \( \lambda \) stands only for the resonant radial and axial mode numbers \( \lambda \) and \( \rho \), respectively. The field co-ordinates \( q_{\lambda \lambda} \) and \( q_{\rho \rho} \) at the \( k \)-th passage of the i-th particle can be computed from the field co-ordinates \( q_{\lambda \lambda} \) and \( q_{\rho \rho} \), calculated at the passage of the previous particle which traversed the cavity at the time \( t - \delta t \). Both can be computed from the equation of a damped oscillator of quality \( Q_i \):

\[ q_{\lambda \lambda} = S_i \exp(-\zeta_i t), \quad q_{\rho \rho} = -\zeta_i q_{\lambda \lambda} + S_i \exp(-\zeta_i t), \quad \text{where} \quad \zeta_i = \omega_i / (2Q_i) \]

\[ S_i = S_0 \cos(\omega_i t) + \frac{1}{\zeta_i} \left[ q_{\rho \rho} - \zeta_i q_{\lambda \lambda} \sin(\omega_i t) \right], \quad q_{\rho \rho} = -\zeta_i q_{\lambda \lambda} + S_i \exp(-\zeta_i t) \]

\[ \Delta B = \frac{(\gamma \delta B_0)}{2} + \frac{(\gamma \delta B_0)}{2} \sin(\omega_i t), \quad \Delta E = \frac{\gamma \delta E_0}{2} \sin(\omega_i t) \]

The pair \( (\xi, E) \) of the next particle can be calculated from the pair \( (\xi, E) \) of the preceding particle by

\[ E_{\lambda \lambda} = E_{\lambda \lambda, k-1} + \frac{\gamma}{2} \sum \frac{1}{\lambda} \left[ q_{\lambda \lambda} \right]_k + \frac{2}{P \lambda} \sin(2\pi f \Delta t) \]

\[ \Delta \lambda = \frac{1}{\lambda} \left[ q_{\lambda \lambda} \right]_k + \frac{2}{P \lambda} \left[ q_{\lambda \lambda} \right]_k \cos(\omega_i t) / \sin(2\pi f \Delta t) \]

where \( P \lambda = \sqrt{1 - \frac{\delta B_0}{\Delta E_0} \frac{2}{\gamma \delta E_0} \frac{\sin(\omega_i t) \cos(\omega_i t)}{\cos(\omega_i t) \sin(\omega_i t)}} \), \( P \lambda \) even

The parameter \( \eta \) describes the characteristics of the focusing magnetic field of an rf-synchrotron. The first term of Eq. (2) is due to the externally applied rf voltage of frequency \( f = \Omega / (2\pi) \) and an integer \( n > 1 \) prevents the possibility of letting the beam interact with an equivalent number of identical cavities. At the beginning of the computer simulation the field co-ordinates \( q_{\lambda \lambda} \) and \( q_{\rho \rho} \) are shifted by the energy content of the cavity may be set equal to zero. The quantity \( P \lambda \) has to be added to \( \Delta \lambda \) each time a particle passes the cavity.

The transformations (1) to (3), for N particles, in the following will be referred to as a "turn". Each of the particles is associated with two pointers, one giving the subscript of the next particle that will arrive at the cavity and one giving the subscript of a previous particle which arrived just before. This cyclic pointer arrangement, named a bidirectional string, has to be modified if a particle overtakes one or several others. In that case only six pointers belonging to five particles have to be changed. The average computer time required on the CDC 7600, is 0.25 microseconds per particle and turn. A detailed description of the computer simulation is found in Ref. 2.

Review of the Linear Theory

In terms of the normalized distribution function \( s_i(w) \), the dispersion relation derived from the Vlasov-equation, after linearization, is

\[ 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial s_i}{\partial w} dw, \quad \text{where} \quad \delta s_i(w) = 1 \]

(4)

The angular velocity \( \dot{\theta} \) of a particle is \( \delta \theta = E/(B^2 \Phi_0) \); and the longitudinal coupling impedance \( Z_0 \) for the cylindrical cavity, in our notation, is

\[ Z_0 = \frac{4\pi B_0^2 \delta E_0}{\delta B_0 \delta E_0} \left[ \frac{1}{(1 + \rho^2) \Phi_0^2 \sin(\rho \Phi_0)} \right] \sin^{(1 - \rho^2)}(\rho \Phi_0) \]

where \( \delta \Phi_0 \) is the impedance of free space. In evaluating the integral of Eq. (4) for
realistic distribution functions, the stability criterion is found to be
\[ \frac{Z}{n} \leq \frac{0.7 \pi R_{E} n_{0} \sqrt{\frac{(x/E)^{2}}{\pi^{2}}}}{2e} \]

The subscript "h" denotes the full width at half height. The growth rates are
\[ \omega_{g} = \frac{n_{0} \sqrt{\pi} \pi^{2}}{2n^{2}} \] and \[ \alpha = \frac{n_{0} \sqrt{\pi} \pi^{2}}{2n^{2}} \]

for a non-energetic beam and for the bag model, respectively. In first approxi-
mation the effective coupling impedance \( Z_{\text{eff}} \) for energetic particles can be assumed to be \( Z_{\text{eff}} = (2\pi \nu)_{\text{th}} \), where \( (2\pi \nu)_{\text{th}} \) is given by the threshold criterion (5).

**Numerical Results and Conclusions**

All numerical results have been obtained for superparticles with a charge to mass ratio of a proton.

**a)** Coasting Beam \((\nu_{0} = 0)\)

The initial values of \((\nu, E_{i})\) were chosen at random so that the azimuthal distribu-
tion is uniform and the energy distribution is Gaussian with initial rms energy spread \(\Delta E_{1/2} \approx \langle E \rangle / \sqrt{8\ln 2} \). As expected, the simulations show that the oscillations of the stored energy of the cavity occur synchronously with the oscillation of the driving Fourier component \(E_{b} \) of the beam. The growth rate has been calculated from fits of an exponential function to the latter in the linear region \((\nu < 1)\). The growth rate is larger than that obtained from linear theory if the damping rate \(\alpha_{e} \) becomes equal to the linear growth rate \(\alpha_{e} \) or larger. If this is not the case, the agreement between our calculations and linear theory is good.

The small-amplitude threshold that determines whether a beam is initially stable, is the same in our simulations as in the linear theory. If the beam is initially unstable, the first growth of the driving beam modulation stops, and it be-
gins to decrease as the moment when the beam width crosses the threshold. This can be verified by Fig. 2 which shows the results as function of the initial spread. The rms energy spread at the first maximum of \(E_{b} \), denoted by \(\Delta E_{1/2}(\tau_{1})\), agrees well with the threshold value \(\Delta E_{1/2}(\nu_{0})\) as given by Eq. (5). The results of our non-linear theory, which were obtained as function of the parameters \(\nu_{0}, \lambda, \eta, G, \eta_{0}, \gamma_{0}, \) and \(\nu_{0}\), agree within a few percent with the small-amplitude threshold.

As Fig. 2 shows, the increase of the energy spread does not stop when the pertur-
bation amplitude and therefore the cavity excitation have reached their first maxima. It converges to a final value \(\Delta E_{r} \) which is estimated by fitting an exponential function to the energy variance. We derive that the final energy spread is inversely pro-
portional to the initial spread. By varying the current, the cavity parameters, and the quantities \(\nu_{0}, \lambda, \eta, G, \eta_{0}, \gamma_{0}, \) we find that the final spread is also proportional to the square of the threshold. These results suggest
\[ \Delta E_{r} = \frac{\Delta E_{1/2}^{2}}{\Delta E_{1/2}} \]

which is plotted in the diagram of Fig. 2.

The parameters obtained from simulations based on only the fundamental \((M_{0})\) mode of the cavity are discussed in more detail in Ref. 26. If the particles interact with several resonant modes of the cavity, the simulations show that the effective coupling impedance determining threshold and growth rate is given by the sum of the indi-
vidual coupling impedances in the case of equal frequencies, and by the dominant impedance in the case where the resonant modes have different frequencies. Again, it has been found that the overshoot is well described by the estimate (6).

**b)** Microwave Instability caused by small Q-factors

If the bunches of a particle beam are formed by an external rf voltage, it is expected that no instabilities occur since, although the disturbances are amplified as they run along the bottom of the bunch, they are equally damped as they run back along the top. On the other hand, it has been conjectured that if a disturbance forms at the back of the bunch and grows to be a finite, large disturbance simply while trav-
elling from the back to the front of the bunch, then the bunch could break up before the damping sets in. Recent observations of bunched beams at CERN has led to this assump-
tion. As these instabilities are associated with high frequency beam oscillations in the range 1 to a few GHz, they have been referred to as microwave instabilities. Because of the rather large uncertainty in the high frequency impedance, the experi-
ments yield no conclusive proof of the hypothesis that the beam instability is due to high frequency wake fields.

In the simulation of the microwave instability an elliptic distribution function of shape \(x(x^{2}+\lambda^{2})^{-3/2} \) was chosen to describe the phase-space density. \(x\) and \(\lambda\) are the azimuthal position and the energy deviation, respectively, with respect to a particle in synchronism with the rf voltage. It can be shown that the quantity in the brackets of Eq. (5) does not depend on \(x\). The growth rate \(\alpha_{e} \), however, is proportional to \(x^{2} \).

The simulations show that in the case of an unstable beam a cluster of high den-
sity, which is formed by particles that get thrown outside of the original bunch area, travels around the bunch center due to the synchrotron motion. After a time which is short compared with the phase oscillation period, the energy variance of the beam reaches a relative maximum, it then decreases slightly until a relative minimum is reached. Subsequently, it increases with fluctuations which can be related to both the synchrotron motion and the instability. The rms energy spread corresponding to the first maximum of the variance has been computed for various impedances. By extrapol-
ating a threshold impedance is found which corresponds within a few percent with the threshold impedance derived from the locally applied criterion (5).

The growth rate of the microwave instability has been estimated by an exponential fit to the variance curves. We found that the growth rate is determined by the current density at the bunch edge. The results obtained agree well with the growth rate \(\omega_{g} \), although in the latter the effect of Landau-damping is not taken into account.

A basic assumption in applying coasting beam theory to bunches is that the wake fields excited in the cavities have a short memory. By several simulations it could be verified that the microwave instability becomes effective only for \(Q < 50-60 \). As it was found in a separate study, the results do not depend very much on the frequency in the GHz-range.

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**Fig. 2** - rms energy spread of the beam as function of the initial spread (in units of the threshold spread).
COLLISIONAL DAMPING OF BERNSTEIN ECHOS

by

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ABSTRACT

The temporal evolution of Bernstein linear wave echoes and Bernstein poles are studied using a Fokker planck collisional model.

A collisional model taking into account both friction and diffusion is used to investigate the behavior of strong phase dependent waves, like linear echoes or Bernstein gyromagnetic waves. Such a model giving an accurate description of the diffusion in the velocity space to characterized electrons-neutral and electrons-ions collisions in a plasma is described by a Fokker Planck equation. A B.G.K. (Bathnagar, Gross, Krook) model describing a Poisson like process, which removes completely from their orbits a fraction of the electrons and leaves the others completely undisturbed, is a poor model to describe the collisional process a problem of wave propagation in a magnetic field. In that case the electrons gyromagnetic radiation is strongly phase dependent and so is more disturbed by a diffusion process than by a Poisson like one.

The Fokker Planck term is written:

$$\frac{\partial f}{\partial t} = \frac{1}{\sqrt{2\pi}} (\vec{V}_p + \vec{W}) \frac{2\pi}{\sigma}$$

(1)

where $\vec{V}_p$ is the thermal speed, $\vec{V}$ is the particles velocity and $\vec{W}$ the collision frequency.

The dielectric constant is now obtained (1):

$$\epsilon = 1 + \frac{1}{\kappa^2 p^2} - \frac{8}{\kappa^2 D^2} \exp \left[ u + n \cos 2\theta \right]$$

(2)

The five independent variables are: $k\delta$ the wave number, $\delta = i\omega$ the complex frequency, $\omega$ the gyrofrequency, $\theta$ the collision frequency and $\theta$ the angle between $\vec{B}$ and $\vec{V}$ (all the frequencies are in plasma frequency, $\omega_p$, units). The result (equation 2) generalises results previously obtained. By Lenard and Bernstein (2) for $\omega = 0$ ; by Gordeev (3) and Bernstein (4) for $\delta = 0$ ; and Dougherty's (5) integral form for the dielectric constant, valid only in the case $\text{Real}(\delta) > (u + n)$, which not apply for Bernstein waves (1).

Equation (2) was now programmed on a digital computer in two forms. The first, for $|n| < 10$, $|u| < 10$, Sums terms two by two until the increment is lower than a definite error. The second, for larger values of $n$ and $u$, which is much slower than the first, stokes terms of the series (still two by two) until the maximum term of the series is obtain and then sums terms of the same magnitude until the condition of convergence is reached.

The roots of equation 2 are found by the downhill method (7).

Coming back to the influence of collisions on waves in a magnetic field, we start with Bernstein echoes. As noticed in a previous paper (6), the answer of a magnetoplasma to an electric field pulse in a direction perpendicular to the magnetic field, results in a succession of echoes, each damped by a phase mixing mechanism, separated by a time $T = 2\pi/\omega_p$, the successive echoes spreading out but their amplitude remains constant. We make $\theta = \pi/2$ and a weak magnetic field characterized by $u_p = .2$.

Summing all the contribution of the seventeen first poles, we obtain the time evolution of the signal and we observe the modification of the excitation spectrum for different values of $\delta$. We have to notice that in the four cases the signal itself is not affected by the collisions while the second echo is totally destroyed for $\delta = 3 \times 10^{-2}$ (figure 1).

As a second example we treat the case of Bernstein's poles for angles not exactly at $\theta = \pi/2$. We find (figure 2) a strong correlation between $\theta$, the smallest angle for which the poles still exist, and the collision frequency $\delta$. In this particular case the poles are strongly damped and so the existence of linear echoes are forbidden, and only strong gryromagnetic waves can propagate at angles $\theta \neq \pi/2$.

To conclude we can say that excitation of these echoes and the study of their destruction would provide a very sensitive measurement of the plasma collision frequency.


Numerical Studies of Particle Confinement in High Beta Equilibrium Mirror Plasmas

BY

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Two codes have been developed to calculate 3D mirror equilibrium and to follow single particle orbits in these equilibrium fields.

In mirror confinement accurate calculation of \( \mu \), the magnetic moment, is essential if we are to predict correct confinement times for the particles studied since \( \mu = \mu_0 \) determines the loss cone boundary. Some conventional single particle orbit codes follow the particle trajectory by finding the magnetic field from interpolation of a gridded representation of \( B \)'s on a 3D mesh. For example, we note the code TIBRO\(^1\), in use at Livermore for over a decade, which uses a cylindrical coordinate grid and in which interpolation is done by the 2nd or 3rd order Lagrange method. In this code, and presumably others, no effort has been made to ensure that the interpolation formulae preserve \( \nabla \cdot B = 0 \). To see how this term is important we refer to the theory of adiabaticity where it is shown\(^2\)

\[
\frac{1}{\mu} \frac{d\mu}{dt} = -V_e \frac{\nabla \cdot B}{B} - \frac{(V_e^2 + 2V_B^2)}{V_e} \left( \frac{\nabla \cdot B}{B} \right) \cos \psi
\]

\[+ \text{higher order physical terms}
\]

The first term on the right side is unphysical but will have a kinematic effect in the code. It is important that this term vanish or at least be very small compared to the other terms. An example of this pathological effect will be presented.

A new code, ORBXYZ\(^2\), has been built to replace TIBRO. It employs a Cartesian 3D grid and it uses vector potentials defined on this grid as the data for the interpolation formulae. A 3D tri-cubic spline interpolation formula is used such that continuity up to the second derivative is maintained at each grid cell interface. The curl is taken analytically from the spline formulae for the \( A \)'s so

\[
\nabla \times A = \nabla \times A
\]

retains the property \( \nabla \cdot B = 0 \) down to computer round off. Thus in the new code ORBXYZ the unwanted effects of finite \( \nabla \cdot B \neq 0 \) are absent.

To use ORBXYZ, methods were developed for placing vector potentials on the 3D mesh. For studies involving vacuum fields the code ABCXYZ\(^4\) is used to calculate the vector potentials of an arbitrary arrangement of current carrying conductors. To obtain high beta equilibrium fields the vacuum fields from ABCXYZ are fed to the vector potential equilibrium code, VEPEC\(^5\), which calculates the fields for realistic pressure models of the form

\[
\rho = \rho(B, \psi).
\]

The equilibrium problem requires the simultaneous solution of Ampere's Law,

\[
\nabla \times A = -\frac{4\pi}{c} J_0,
\]

(4)

(together with the pressure balance equation

\[
\nabla \cdot \rho = \frac{1}{c} J \times B.
\]

Charge neutrality is assumed and we use the Coulomb gauge \( \nabla \cdot A = 0 \). To prevent accumulation of charge

\[
\nabla \cdot \rho = 0
\]

(6)

is included. Other constraints restricting the shapes of the flux surfaces

[1] TIBRO

[2] ORBXYZ

[3] ABCXYZ

[4] VEPEC
References


\[ \psi \text{ enter because the normal current } J_n \text{ must vanish everywhere on the plasma surface. From the equilibrium theory of Hall and McNamara\textsuperscript{6} these extra constraints are expressed as magnetic differential equations} \]

\[ \nabla \cdot (\mathbf{B}) = \hat{\rho}_n(B) \quad (7) \]

\[ \nabla \cdot (\psi \mathbf{B}) = 0. \quad (8) \]

Here we are assuming the pressure model has the product form

\[ P_n(B, \psi) = \hat{\rho}_n(B) \psi \quad (9) \]

and

\[ \psi = \Gamma \quad (10) \]

on some surface beyond the plasma and perpendicular to the flux lines.

Numerical solution of the above equations is obtained iteratively by a relaxation method. Eq. 4 is solved by the Douglas-Gunn algorithm\textsuperscript{7} which is similar to ADI. At every iteration \( \psi \) is determined again from Eqs. (7) and (8). \( J_1 \) and \( J_n \) are found from Eqs. (5) and (6) respectively thus updating the right side of (4). A typical high beta equilibrium solution requires about 100 iterations or approximately 10 minutes on the CDC 7600.

Results showing plasma \( B \approx 0.5 \) will be shown together with orbits calculated for these high beta fields.
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