Trapped Ion Depletion by Anomalous Diffusion Due to the Dissipative Trapped Ion Instability

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Abstract

At high temperatures the KADOMTSEV - POGUTSE diffusion in tokamaks can become so large as to cause depletion of trapped ions if these are replaced with free ions by means of collisions rather than being directly recycled or injected. Modified KADOMTSEV - POGUTSE diffusion formulas are employed in order to estimate this effect in the cases of classical and anomalous collisions. The maximum trapped-ion depletion is estimated from the PENROSE stability condition. For anomalous collisions a BOHM - type diffusion is derived. Numerical examples are given for JET-like parameters (JET = Joint European Torus). Depletion is found to reduce diffusion by factors of up to 10 and more.
1. INTRODUCTION

In 1970 KADOMTSEV and POGUTSE published their well-known formula of the anomalous diffusion coefficient due to the dissipative trapped-ion instability (see KADOMTSEV and POGUTSE 1970, 1971). This formula has been applied to experimental conditions in tokamaks by FURTH (1973), PFIRSCHE et al. (1974), and GREEN et al. (1974). SAISON and WIMMEL (1975) extended the diffusion formula to plasmas with an ion charge \( Ze \), \( Z > 1 \), with the result that the diffusion coefficient drops approximately in proportion to \( 1/Z^3 \) for not too large \( Z \). KADOMTSEV and POGUTSE (1970, 1971) derive their diffusion formula from dissipative continuity equations for trapped ions and electrons together with the quasi-neutrality condition, in which certain approximate expressions for the free-particle densities appear.

In deriving their diffusion formula KADOMTSEV and POGUTSE assumed that the average trapped-ion and electron densities are not affected by the anomalous diffusion process. It is this assumption that I want to replace by a more realistic one. A simple estimate shows that, at high temperatures, the anomalous diffusion can cause depletion of the trapped ions if these are only replaced with free ions by means of collisions - COULOMB or anomalous - rather than being injected directly from outside. It will turn out that for a small degree of depletion (at moderately high temperatures) the collisions are of the classical COULOMB type.
For depletions larger than a critical value (at still higher temperatures) an inverse loss-cone instability develops that leads to anomalous collisions. The two cases will be studied separately.

The question exactly what set of basic equations should be used in treating the effect of trapped-ion depletion deserves some special attention for the following reason. An improved set of trapped-fluid equations has been established by the present author (WIMMEL 1975) with corrected expressions for the free-particle densities and the collision terms, and with additional source terms in the continuity equations. In particular, source terms coming from $\nabla \delta_0 (\delta_0 =$ equilibrium fraction of trapped particles) have been taken into account, and the corrected equations show that the anomalous trapped-ion diffusion depends in sign on the total density gradient rather than the trapped-density gradient. This is important near the magnetic axis of the plasma torus. Apart from this effect, the new equations yield diffusion fluxes of similar magnitude to those of the original KADOMTSEV - POGUTSE equations for such values of the aspect ratio as occur in practice. Only in the mathematical limit of very large aspect ratios does the correction factor for the diffusion flux become very small compared with unity. For this reason a corrected version of the KADOMTSEV - POGUTSE equations that includes the $\nabla \delta_0$ correction will be used as the starting point of these calculations.
Section 2 presents the basic trapped-fluid equations and the general expressions for the anomalous trapped-particle diffusion flux. Section 3 gives the formulas for depleted transport with COULOMB collisions. In Section 4 the inverse loss-cone instability leading to anomalous collisions is studied by means of the PENROSE stability condition. Section 5 gives the formulas for depleted transport with anomalous collisions. In Section 6 the limiting temperatures separating the different diffusion regimes are derived. Section 7 presents numerical examples of the resulting anomalous diffusion as a function of temperature. Section 8 completes the study with a summary and discussion.

2. BASIC EQUATIONS AND TRANSPORT

The trapped-fluid equations in two spatial dimensions, as introduced by KADOMTSEV and POGUTSE (1970, 1971) but supplemented by a \( \nabla \phi \) correction term, read:

\[
\frac{\partial n_i}{\partial t} + \mathbf{v} \cdot \nabla n_i = -\nu_{\text{eff}}(n_i - n_e) + N_p \mathbf{v} \cdot \nabla \phi
\]  

\( (2.1) \)

\[
\frac{\partial n_e}{\partial t} + \mathbf{v} \cdot \nabla n_e = -\nu_{\text{eff}}(n_e - n_0) + N_p \mathbf{v} \cdot \nabla \phi
\]  

\( (2.2) \)

\[
\phi = \left( \frac{T}{2eN_p} \right) (n_i - n_e)
\]  

\( (2.3) \)

\[
\mathbf{v} = \left( cT/2eBN_p \right) \hat{z} \times \nabla (n_i - n_e)
\]  

\( (2.4) \)
Here $n_i$, $n_e$ are the trapped-particle densities, $\mathbf{v}$ is the common $E \times B$ drift velocity, $\nu_{\text{eff}}$, $\nu_{\text{eff}}$ are the effective collision frequencies of the trapped particles, $\Phi$ is the electric potential, $N_i = \sum_o^p N_p$ and $N_p$ are the trapped and total particle densities in equilibrium; the remaining notation is standard, except for $\mathbf{T} = \frac{2T_e T_i}{(T_e + T_i)}$. It is assumed that $\nabla \mathbf{B} = \nabla (T/N_p) = 0$, hence $\nabla \cdot \mathbf{v} = 0$. A Cartesian slab model, with the coordinates $x$, $y$, will be used throughout.

In an earlier paper (Wimmel 1975, henceforth called (1)) it was shown that the anomalous trapped-particle diffusion following from eqs. (2.1) to (2.4) and periodic boundary conditions in $y$ is ambipolar and obeys the general relation for the flux density:

$$\mathbf{j}_x = -\left( \frac{cT}{2a B N_p \nu_{\text{eff}}} \right) \left< \partial_x \mathbf{p} \cdot \partial_y \mathbf{p} \right> ,$$

with $\mathbf{p} = n_i \mathbf{v}_i - n_e \mathbf{v}_e$, the pointed brackets denoting a time average. In terms of critical FOURIER modes the random-phase approximation gave the estimate

$$\mathbf{j}_x = \frac{A \nu_{\text{eff}}}{\nu_{\text{eff}}^2 + \omega_k^2} \left< |\mathbf{n}_{iK}|^2 \right> ,$$

with $Z_k$ = number of critical modes of common wave vector $K$ and frequency $\omega_k$; $\mathbf{n}_{iK}$ = FOURIER transform of $(n_i - \bar{n}_i)$, $\bar{n}_i$ being the time-averaged $n_i$; and $A = cT/2e B N_p$. In (1) the mean square amplitude of $\mathbf{n}_{iK}$ was estimated by the mixing-length hypothesis in the
form of
\[ Z_K k_x^2 \left< \hat{n}_{iK} \right|^2 \approx |\nabla n_i|^2, \]  
(2.7)

with \( k_x^2 = k_y^2 \). In the present case of trapped ion depletion we make
instead the plausible ansatz:
\[ Z_K k_x^2 \left< \hat{n}_{iK} \right|^2 \approx |\nabla \bar{n}_i|^2, \]  
(2.8)

again with \( k_x^2 = k_y^2 \). This leads to the smaller flux density
\[ J_x = \frac{A \nu_{\text{eff}} \omega_K}{k_y (\nu_{\text{eff}}^2 + \omega_K^2)} |\nabla \bar{n}_i|^2. \]  
(2.9)

Let us turn to determining the critical mode parameters \( K_y \) and \( \omega_K \). In
(I) it was assumed that \( \omega_K \) obeys the linear dispersion equation \( \omega_K = \omega(K_y) \)
associated with eqs. (2.1) to (2.4), where \( \omega \) is the real part of the
complex frequency. In the present case one may surmise that the trapped
ion depletion, with \( \bar{n}_i < n_0 \), leads to smaller effective values of
\( \omega_K(K_y) \) than given by the linear dispersion equation. For simplicity,
the linear dispersion equation will nevertheless be used. In this way
an upper bound for the flux density \( J_x \) will be obtained.

The dispersion equation is given by (see (I))
\[ -i\omega + y = -\frac{1}{2} \left\{ 1 \pm \left( 1 + \frac{4i \omega \nu_d - \nu_{\text{eff}} \nu_{\text{eff}}^*}{\nu_5^2} \right)^{1/2} \right\}, \]  
(2.10)
of which we use the values of the unstable modes. Here \( \omega_n = K_y v_n \),
\( v_n = A \delta_0 \partial_x N_p \), \( v_S = \frac{1}{2} (v_{eff} + v_{ref}) \), \( v_d = \frac{1}{4} (v_{eff} - v_{ref}) \),
\( \omega \) is the (real) frequency, and \( v \) is the growth rate. The critical \( \omega_k \)
in the case of small-scale turbulence \(|K_y| a > \pi\) and the critical \( K_y \)
in the case of large-scale turbulence \(|K_y| a \sim \pi\), where \( a \) is the small
plasma radius, are determined as in (I). In this way, for small-scale
turbulence the flux density becomes

\[
y_x = -\frac{A^2}{2 \sqrt{5} \nu_{eff}} \left| \nabla n_i \right|^2 \partial_x N_p ,
\]

\( (2.11) \)

with

\[
|K_{y \text{crit}}| = \frac{\sqrt{5} \nu_{eff}}{|v_n|} .
\]

\( (2.12) \)

For large-scale turbulence one obtains instead

\[
y_x = -\left( \frac{\alpha}{\pi | \partial_x N_p |} \right)^{\frac{3}{2}} \left| \nabla n_i \right|^2 \left( \frac{2A \nu_{eff}}{\delta_0} \right)^{\frac{1}{2}} \partial_x N_p ,
\]

\( (2.13) \)

with

\[
|\omega_{\text{crit}}| = \left( \frac{\pi}{2} \frac{\nu_{eff} |v_n|}{\alpha} \right)^{\frac{1}{2}} .
\]

\( (2.14) \)

In order to determine \( \bar{n}_i \) in eqs. \((2.11), (2.13)\), the particle balance
equation of trapped ions must be used in addition. This balance equation
is obtained by time-averaging and integrating over \( x \) eq. (2.1),

whence

\[
y_x' \equiv \langle n_i \cdot v_x' \rangle = \int_0^x d\xi' \nu_\text{eff} (\bar{n}_o - \bar{n}_i)
\]  

(2.15)

Here the term involving \( \nabla \delta_o \) disappears owing to \( \langle v_x' \rangle = 0 \).

The effects of trapped-ion depletion will be estimated in the sections to follow by means of a zero-dimensional analysis. This means that I shall put \( \partial_x \to 1/a \), where \( a \) is the minor plasma radius. Therefore eq. (2.15) will be used in the form

\[
|y_x'| = \alpha \nu_\text{eff} (\bar{n}_o - \bar{n}_i)
\]  

(2.16)

while eqs. (2.11) and (2.13) will be used in the forms

\[
|y_x'| = \frac{A^2 \bar{n}_i^2 \bar{n}_o}{2\pi \nu_\text{eff}} a^3
\]  

(2.17)

and

\[
|y_x'| = \pi^{3/2} \bar{n}_i^2 \left( \frac{2 \nu_\text{eff} A}{\bar{n}_o} \right)^{1/2}
\]  

(2.18)

In plasma regimes where anomalous collisions are effective \( \nu_\text{eff} \) and \( \nu_\text{eff} \) will be replaced by anomalous values. In addition, for the numerical examples to be considered, a characteristic Lawson quantity

\[
\nu_\tau = a N_p^2 / |y_x'|
\]  

(2.19)
will be plotted as a function of temperature (and other plasma parameters) rather than the flux density \( \gamma_x \) itself. This will make an assessment of the numerical results in terms of experimental performance easier.

3. DEPLETED TRANSPORT WITH COULOMB COLLISIONS

In the case of small-scale turbulence the diffusion is determined by eqs. (2.16) and (2.17). Hence, putting \( \bar{N}_i = \delta_i N_p \), \( \gamma_x \) and \( n \tau \) are determined by

\[
|\gamma_x| = \alpha \nu_{\text{eff}} N_p (\delta_0 - \delta_i) = \frac{A^2 \delta_0 N_p^3}{2 \sqrt{5} a^3 \nu_{\text{eff}}} \delta_i^2 \tag{3.1}
\]

and

\[
n \tau = \frac{N_p}{\nu_{\text{eff}} (\delta_0 - \delta_i)} = \frac{2 \sqrt{5} a^4 \nu_{\text{eff}}}{A^2 N_p^2 \delta_0 \delta_i^2} \tag{3.2}
\]

with \( \delta_i \) satisfying the quadratic equation

\[
\delta_i^2 + K_\alpha (\delta_i - \delta_0) = 0, \tag{3.3}
\]

where

\[
K_\alpha = \frac{2 \sqrt{5} a^4 \nu_{\text{eff}} \nu_{\text{eff}}}{A^2 N_p^2 \delta_0} \tag{3.4}
\]
For large values of \( K_1 \) the solution of eq. (3.3) approaches the equilibrium value, \( \delta_i \rightarrow \delta_o \).

In the case of large-scale turbulence the diffusion is determined by eqs. (2.16) and (2.18). Hence

\[
|y'\alpha'| = a \nu_{\text{eff}} N_p (\delta_o - \delta_i) = \left( \frac{N_p}{\pi} \right)^{\frac{3}{2}} \left( \frac{2 A \nu_{\text{eff}}}{\delta_o} \right)^{\frac{1}{2}} \delta_i \frac{\delta_i^2}{\nu_{\text{eff}}} \tag{3.5}
\]

and

\[
\nu = \frac{N_p}{\nu_{\text{eff}} (\delta_o - \delta_i)} = \left( \frac{\delta_o N_p}{2 A \nu_{\text{eff}}} \right)^{\frac{1}{2}} \frac{\pi^2 a}{\delta_i^2} \tag{3.6}
\]

with \( \delta_i \) determined by

\[
\delta_i^2 + K_2 (\delta_i - \delta_o) = 0, \tag{3.7}
\]

where

\[
K_2 = \pi^{3/2} a \nu_{\text{eff}} \left( \frac{\delta_o}{2 A N_p \nu_{\text{eff}}} \right)^{\frac{1}{2}}. \tag{3.8}
\]

Again, for large values of \( K_2 \) the equilibrium solution of \( \delta \) is approached, \( \delta_i \rightarrow \delta_o \).

The results of this section hold in the absence of anomalous collisions, i.e. for trapped-ion densities greater than the critical density given in the next section, eq. (4.14).
4. INVERSE LOSS-CONE INSTABILITY

Only for trapped-ion depletions that are not too large do the collisions of ions and electrons remain classical. For larger depletions an inverse loss-cone instability develops that will provide for fluctuating electric fields and anomalous collisions. This instability will be referred to as "inverse loss-cone" because here the "loss-cones" actually remain filled with circulating particles, while the complementary volume in velocity space becomes partially depleted of trapped particles. In the following, the critical depletion for the onset of the instability will be estimated from the PENROSE criterion (PENROSE 1960, McCUNE 1965). A rough approximation for the ion distribution function will be employed and only perturbations with $k \parallel B_o$ will be considered because for other directions of $k$ the PENROSE criterion does not apply in the presence of $B_o \neq 0$.

The PENROSE criterion yields instability if and only if

$$I = : P \int_{-\infty}^{+\infty} \frac{du}{u-u_o} \frac{F'(u)}{F(u)} > 0 , \quad (4.1)$$

with

$$F(u) = \sum_{i,e} \omega_{pi}^2 \int d^2v_\perp f_\mu(v_\perp, u) \quad (4.2)$$

and $F'(u_o) = 0$. Because of symmetry we have $u_o = 0$. Putting $F = \frac{F_i}{e} + \frac{F_e}{e}$, $I = I_i + I_e$, a Maxwell distribution of electrons (no depletion) yields
\[ F_e(u) = \omega_{pe}^2 \left( \frac{m_e}{2\pi T_e} \right)^{\frac{1}{2}} e^{-\frac{m_e u^2}{2T_e}} \]  

(4.3)

and

\[ I_e = -4\pi N_p e^2 / T_e. \]  

(4.4)

For the ions we make the ansatz of a step function in velocity space, with overpopulation in the loss-cone region (circulating ions) and underpopulation for the trapped ions, viz.

\[ f_i \, d^2 \xi_\perp \, d\eta = \frac{K}{\pi^{3/2}} e^{-\left(\xi_\perp^2 + \eta^2\right)} \left[ H\left(|\eta| \tan \varphi - \xi_\perp\right) \right. \]

\[ + \left. \Delta \cdot H\left(\xi_\perp - |\eta| \tan \varphi\right) \right] d^2 \xi_\perp \, d\eta. \]  

(4.5)

Here \( H \) is the HEAVISIDE step function, and

\[ \xi_\perp^2 = \frac{m_i}{2T_i} u_\perp^2, \quad \eta^2 = \frac{m_i}{2T_i} u^2, \]  

(4.6)

\[ \cos \varphi = \delta_0, \quad \sin \varphi \geq 0, \]  

(4.7)

\[ K = \left(1 - \delta_0\right) / \left(1 - \delta_i\right), \]  

(4.8)

\[ \Delta \cdot K = \delta_i / \delta_0, \]  

(4.9)

hence \( f_i \) satisfies the relations

\[ \int_{\text{total}} f_i \, d^2 \xi_\perp \, d\eta = 1, \quad \int_{\text{trapped}} f_i \, d^2 \xi_\perp \, d\eta = \delta_i. \]  

(4.10)
It follows that
\[ F_i(\eta) \, d\eta = \frac{\omega_{pi}^2}{\sqrt{\pi}} K \left\{ e^{-\eta^2} - \left(1 - \Delta \right) e^{-\eta^2/\delta_0^2} \right\} \, d\eta \quad (4.11) \]

and
\[ \Gamma_i = -4\pi N_p e^2 \frac{S}{T_i} \quad (4.12) \]

with
\[ S = \left( \delta_i - \delta_o + \delta_i^2 \delta_o \right) / \delta_o^2 \quad (4.13) \]

Evaluating the PENROSE condition \( \Gamma > 0 \) yields instability for depletions \( \delta_i < \delta_c \), with
\[ \delta_c = \frac{\delta_o}{1 + \delta_o} \left(1 - \delta_o \frac{T_i}{T_e} \right) \quad (4.14) \]

5. DEPLETED TRANSPORT WITH ANOMALOUS COLLISIONS

For overcritical depletion of trapped ions, i.e. for \( \delta_i < \delta_c \) [see eq. (4.14)], anomalous collisions of ions and electrons will be produced by the inverse loss-cone instability. The anomalous effective collision frequencies of trapped ions and electrons will be called \( \nu_{ifan}^{i} \) and \( \nu_{efan}^{e} \), respectively. A rough estimate of them is gained by assuming BROWNIAN motion in velocity space with \( K^{\nu_{ifan}^{i}} \gtrsim \omega \) and \( K^{\nu_{efan}^{e}} \sim 1 \), viz.
\[ \nu_{ifan}^{i} \sim \frac{8 \pi^2}{\delta_o^2} \propto \omega_{pi}^2 \left( T_e / T_i \right)^{3/2} \quad (5.1) \]
\[ \nu_{\text{fan}} \sim \frac{8 \pi^2}{\sigma_0^2} \alpha \omega_{pe} \]  

(5.2)

where \( \alpha = E^2/8\pi N_p T_e \). Putting, for example, \( \alpha \sim 10^{-2} \) would lead to \( \nu_{\text{fan}} \sim \omega_{pi} \gg \nu_{\text{eff}} \). This shows that \( \alpha \) will easily adjust itself in such a way that the trapped-ion density is prevented from dropping below the critical density as determined by eq. (4.14). Hence one is justified in putting \( \delta_i \approx \delta_c \) and determining the anomalous collision frequency \( \nu_{\text{fan}} \) from the trapped-ion particle balance. In addition, it is seen that

\[ \frac{\nu_{\text{fan}}}{\nu_{\text{eff}}} = \left( \frac{m_e/m_i} \right)^{1/2} \left( \frac{T_e}{T_i} \right)^{\frac{3}{2}} \]  

(5.3)

The following procedure is now followed in order to compute the depleted diffusion in the regime of anomalous collisions. In eqs. (2.16) to (2.18) the quantities \( \nu_{\text{eff}} \) and \( \nu_{\text{eff}} \) are replaced by \( \nu_{\text{fan}} \) and \( \nu_{\text{fan}} \). Then \( \nu_{\text{fan}} \) is eliminated by means of eq. (5.3). After that, eq. (2.16) together with eq. (2.17), or (2.18), provides a system of 2 equations for the unknowns \( \nu_{\text{hi}} \) and \( \nu_{\text{fan}} \), while \( \tilde{\nu}_{\text{i}} \) is fixed by \( \tilde{\nu}_{\text{i}} = m_c = \frac{Q_c N_p}{}. \) One notices that this approximate procedure is largely independent of the specific instability responsible for the anomalous collisions, as long as the relations \( K \nu_{\text{hi}} \geq \omega \), \( K \lambda_b \sim 1 \) are satisfied; with the exception that the critical depletion \( \delta_c \) was determined on the basis of a specific ion distribution function roughly describing trapped-ion depletion.

For small-scale turbulence the results are as follows:
\[ |j_x| = \alpha \nu_{\text{fan}} \, N_p \, (\delta_0 - \delta_c) \]
\[ = \left( \frac{m_e}{5 m_i} \right)^{1/4} \left( \frac{T_e}{T_i} \right)^{3/4} \left( \delta_0 (\delta_0 - \delta_c) \right)^{1/2} \frac{\delta_c N_p \, c \, T}{2 \, \alpha \, e \, B} \tag{5.4} \]

and

\[ n_T = \frac{N_p}{\nu_{\text{fan}} (\delta_0 - \delta_c)} \]
\[ = \left( \frac{5 m_i}{m_e} \right)^{1/4} \left( \frac{T_i}{T_e} \right)^{3/4} \left( \frac{2}{\delta_0 (\delta_0 - \delta_c)} \right)^{1/2} \frac{2 \, e \, B \, N_p \, a^2}{c \, T \, \delta_c} \tag{5.5} \]

with

\[ \nu_{\text{fan}} = \left( \frac{m_e}{5 m_i} \right)^{1/4} \left( \frac{T_e}{T_i} \right)^{3/4} \left( \frac{\delta_0}{2 (\delta_0 - \delta_c)} \right)^{1/2} \frac{\delta_c \, c \, T}{2 \, \alpha^2 \, e \, B} \tag{5.6} \]

and \( \delta_c \) given by eq. (4.14). As is seen from eqs. (5.4), (5.5), this is BOHM-type diffusion.

For large-scale turbulence one has instead

\[ |j_x| = \alpha \nu_{\text{fan}} \, N_p \, (\delta_0 - \delta_c) \]
\[ = \frac{1}{\pi^3} \left( \frac{m_i}{m_e} \right)^{1/2} \left( \frac{T_i}{T_e} \right)^{3/2} \frac{\delta_c^4}{(\delta_0 - \delta_c) \delta_0} \frac{c \, T \, N_p}{\alpha \, e \, B} \tag{5.7} \]

and
\[ nT = \frac{N_p}{\nu_{ifan} (\delta_0 - \delta_c)} \]

\[ = \pi^3 \left( \frac{m_e}{m_i} \right)^{1/2} \left( \frac{T_e}{T_i} \right)^{3/2} \frac{\delta_0 (\delta_0 - \delta_c)}{\delta_c^4} \frac{a_i^2 e B N_e}{c T} \]  \hspace{1cm} (5.8)

with a different value of \( \nu_{ifan} \), viz.

\[ \nu_{ifan} = \frac{1}{\pi^3} \left( \frac{m_i}{m_e} \right)^{1/2} \left( \frac{T_i}{T_e} \right)^{3/2} \frac{\delta_c^4}{\delta_0 (\delta_0 - \delta_c)^2} \frac{c T}{a_i^2 e B} \]  \hspace{1cm} (5.9)

This again is BOHM-type diffusion.

6. LIMITING TEMPERATURES

In Sections 3 and 5 four different diffusion formulas have been given for the cases of small-scale and large-scale turbulence, with COULOMB and anomalous collisions, respectively. In order to apply these formulas, their ranges of validity in parameter space must be known. For this purpose the limiting temperatures of the four ranges are given in the following. Which of these are applicable for a specific plasma will be determined for the numerical examples considered in the next section. Because the limiting temperatures follow by elementary calculations from the formulas of Sections 3, 4, and 5, the results are simply listed.

For small-scale turbulence the limiting temperature between the ranges of COULOMB collisions and anomalous collisions is found from the
condition $\delta_i = \delta_c$ to be

$$T_{e1} = \frac{4}{3} \left( \frac{a Z_{eff}}{\delta_o} \right)^{4/5} \left[ 10^{-22} \frac{e B N_p}{c \delta_c} \left( 1 + \frac{T_e}{T_i} \right) \right]^{2/5} \left( \frac{T_e}{T_i} \right)^{3/10} \left[ 2 \sqrt{5} (\delta_o - \delta_c) \right]^{1/5} \left( \frac{m_e}{m_i} \right)^{1/10}$$

(6.1)

where the following expressions for $\nu_{eff}$ and $\nu_{iff}$ have been used:

$$\nu_{eff} = 10^{-22} N_p Z_{eff} / T_e^{3/2} \delta_o^2 \quad (6.2)$$

$$\nu_{iff} = 10^{-22} (N_p Z_{eff}^3 / T_i^{3/2} \delta_o^2) \left( m_e / m_i \right)^{1/2} \quad (6.3)$$

All formulas are in c.g.s. units. For large-scale turbulence one has instead

$$T_{e2} = \frac{Z_{eff}^2 (\pi T_e)}{T_i} \left[ \frac{a (\delta_o - \delta_c)}{\delta_c^2} \right]^{4/5} \left[ 10^{-22} \frac{e B N_p}{2 c \delta_o} \frac{m_e}{m_i} \left( 1 + \frac{T_e}{T_i} \right) \right]^{2/5}$$

(6.4)

as the limit between the ranges of COULOMB and anomalous collisions.

In the range of COULOMB collisions the limiting temperature between the ranges of small-scale and large-scale turbulence is obtained, from the condition $|k_y^{crit}| = \pi / a$, as:
\[ T_{e3} = \left[ \frac{V_S}{\pi} 10^{-22} \frac{Z_{\text{eff}}}{c} \frac{eB}{\delta_o} N_p a^2 (1 + \frac{T_e}{T_i}) \right]^{2/5} \] (6.5)

On the contrary, in the range of anomalous collisions a limiting temperature \( T_{e4} \) between the ranges of small-scale and large-scale turbulence does not exist. Independent of \( T_e \), the turbulence is small-scale or large-scale for

\[ f = \frac{\pi}{5} \left[ 2 \delta_o (\delta_o - \delta_c) \right]^{1/2} \left( \frac{m_e}{5 m_i} \right)^{1/4} \left( \frac{T_e}{T_i} \right)^{3/4} \leq 1 \] (6.6)

respectively.

In addition, two more characteristic temperatures will be given that determine whether certain necessary conditions for the validity of the diffusion formulas are satisfied. Neglecting any effects of LANDAU damping, one necessary validity condition requires that the critical wavelength be greater than an ion gyroradius. For smaller critical wavelengths the drift approximation breaks down. This is satisfied for

\[ T_e > T_{e5} = \left( \frac{10 m_i T_i}{T_e} \right)^{1/4} \left[ 10^{-22} \frac{a N_p Z_{\text{eff}}}{\delta_o^3} \left( 1 + \frac{T_e}{T_i} \right) \right]^{1/2} \] (6.7)

This limiting temperature has been evaluated only for the case of small-scale turbulence with COULOMB collisions. This suffices for the
numerical examples considered in the next section. Another necessary
validity condition requires that the frequency of the critical modes be
smaller than the transiting frequency of the untrapped ions with respect
to the minor circumference of the torus. One reason for this requirement
is to justify the unequal treatment of trapped and untrapped particles
in the basic eqs. (2.1) to (2.4). The condition is satisfied for

\[ T_e > T_{e6} = \left( \frac{m_i T_i}{3 T_e} \right)^{1/4} \left[ 10^{-22} \frac{Rq N_p Z_{eff}}{\delta_o^3} \right]^{1/2}, \quad (6.8) \]

where \( R \) is the major torus radius, and \( q = 2 \pi / \lambda \) is the usual safety
factor. Again, this limiting temperature has been evaluated only for
small-scale turbulence with COULOMB collisions because this is sufficient
for the numerical examples to follow.

7. NUMERICAL EXAMPLES

In order to illustrate the practical consequences of the above
formulas, I shall present two numerical examples. The parameters used
are as follows: \( N_p = 10^{13} \) and \( 10^{14} \text{ cm}^{-3} \), \( B = 3 \times 10^4 \text{ G} \), \( T_i = T_e \),
\( Z_{eff} = 1 \), \( a = 100 \text{ cm} \), \( R = 300 \text{ cm} \), \( q = 2.5 \), \( \delta_o = 0.5 \). The functions
plotted are \( \mathcal{M}(T_e) \); see Figs. 1 and 2.

In order to construct the graphs of Figs. 1 and 2, one must first
determine which of the four formulas of eqs. (3.2), (3.6), (5.5), (5.8)
apply in what temperature range. Hence the critical temperatures \( T_{e1}, T_{e2}, T_{e3} \), and the quantity \( f \) must be evaluated. For the case \( \mathcal{N}_p = 10^{13} \text{ cm}^{-3} \) one obtains \( T_{e1} = 12.0 \text{ keV}, T_{e2} = 6.65 \text{ keV}, T_{e3} = 12.4 \text{ keV}, f = 0.929 < 1 \). It follows that for \( T_e < T_{e1} \) small-scale turbulence with COULOMB collisions obtains, while for \( T_e > T_{e1} \) small-scale turbulence with anomalous collisions obtains. The critical temperatures \( T_{e2} \) and \( T_{e3} \) do not have physical significance here because the conditions for their validity are not satisfied in the pertinent temperature ranges. For the case \( \mathcal{N}_p = 10^{14} \text{ cm}^{-3} \) the discussion is analogous, the critical values being \( T_{e1} = 30.1 \text{ keV}, T_{e2} = 16.7 \text{ keV}, T_{e3} = 31.1 \text{ keV}, f = 0.929 \).

Figures 1 and 2 each show three diffusion regimes, viz. undepleted KADOMTSEV-POGUTSE diffusion at low temperatures, with COULOMB collisions effective; then depleted KADOMTSEV-POGUTSE diffusion at higher temperatures, still with COULOMB collisions in effect; and, finally, depleted BOHM-like diffusion, with anomalous collisions. The extrapolated curves (broken lines) show, first, that for \( T_e \approx T_{e1} \) the effect of depletion reduces the particle losses by a factor of approximately 10 in the examples considered, and, second, that without anomalous collisions, \( \mathcal{N}_T(T_e) \) would show a minimum value near \( T_e = T_{e1} \) and increase again for larger temperatures, \( T_e > T_{e1} \). In addition to \( T_{e1} \), the critical temperatures \( T_{e5} \) and \( T_{e6} \) have been indicated, showing that the corresponding necessary validity conditions of the theory are satisfied in the temperature range of interest.
8. SUMMARY AND DISCUSSION

It has been shown that at high temperatures the anomalous diffusion due to the dissipative trapped-ion instability causes depletion of trapped ions if these are not directly replaced by injection from outside. This depletion leads to lower diffusion than that according to the formula of KADOMTSEV and POGUTSE. At high depletion an inverse loss-cone instability develops and induces anomalous collisions. In this regime a BOHM-like diffusion law is obtained. The numerical examples presented in Section 7 show that under experimental conditions of current interest the depletion effect is potentially important because it leads to reductions of the diffusion by a factor of 10 and more.

In keeping with the original theory by KADOMTSEV and POGUTSE only the case of ions with $Z = 1$ and no impurities present has been considered. As is known from previous work (SAISON and WIMMEL, 1975; DOBROWOLNY, 1974), a multiple ion charge $Z \varepsilon$, with $Z > 1$, or the presence of impurities with $Z > 1$ may reduce the anomalous diffusion considerably. Under such conditions, therefore, the depletion of trapped ions usually will not be a large effect. This justifies the restriction to $Z = 1$ in the present study. As has been mentioned elsewhere (WIMMEL, 1975), BOHM diffusion due to the collisionless trapped-ion instability may be superimposed on the regimes of large-scale turbulence depending
on whether the former can be stabilized or not. In the numerical examples considered above this problem does not arise because only small-scale turbulence occurs there.

The above analysis of the depletion effect is zero-dimensional. Generalizing to a 1-dimensional analysis will change the particle balance equation, eq. (2.16), into a differential equation, with the flux $\gamma^\infty_1$ expressed by density gradients. A solution in closed analytical form is then impossible. Rather a numerical transport code must be used. Also the untrapped particles must then be described in a selfconsistent way, and the use of the necessary boundary conditions will require that external particle sources, charge exchange, and additional diffusion mechanisms be included in the code. It may even be necessary to use spatially averaged gradients in the computation in order to represent the nonlocal character of the turbulent diffusion. It appears that such a program would have its main interest in the application to specific machine studies. In order to obtain a general result in the way of an order-of-magnitude estimate, the present zero-dimensional treatment clearly is more appropriate.

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REFERENCES


FIGURE CAPTIONS

Fig. 1  The function $\nu T_e(T_e)$ for JET-like parameters with $N_p = 10^{13}$ cm$^{-3}$.

Fig. 2  The function $\nu T_e(T_e)$ for JET-like parameters with $N_p = 10^{14}$ cm$^{-3}$.
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