Laser Energy Necessary for Inertially Confined Nuclear Fusion Plasma

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ABSTRACT:

The gain $G$ of the thermonuclear fusion energy of very dense spherical plasmas heated by lasers in times of about 1 nsec is treated theoretically. Assuming as a pessimistic estimate that fast ions leave the plasma without contributing to fusion neutrons, we find that for $G = 1$ laser energies of $3 \times 10^7$ joules are needed. A kinetic theory for a one species gas starting with a Gaussian density and velocity profile yields almost the same result as found by a hydrodynamic model with adiabatic expansion and without fast ion losses taken into account. Therefore, the even-point of $G = 1$ is reached at the well-known value for the laser energy of $4.6 \times 10^6$ joules for a D-T reaction.

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I. Introduction

Two different ways of attaining thermonuclear conditions are being studied at present: one is to heat a low-density plasma rather slowly and confine it sufficiently long by magnetic fields; the other is to heat a high-density plasma very rapidly, which then expands freely, this being called inertial confinement. The latter might be possible by using laser pulses, as is discussed here, or, by means of intense electron beams. This paper continues the discussion of some theoretical aspects that was started some years ago to calculate the energies necessary to reach the even-point of energy production, i.e. where energy production equals energy input. In particular, the influence of fast ion losses is discussed.

Two models will be considered. In the first one the dynamics are described by a hydrodynamic theory, while for fusion reactions certain assumptions about the loss of fast ions are made. These assumptions are probably too pessimistic, as can be seen from our second model, in which the dynamics are described by a one species kinetic theory including collisions. The time dependent problem with initial Gaussian density profile and a Gaussian velocity distribution is solved exactly. The fast ion losses are then automatically taken into account. It turns out that the result for the fusion gains is exactly the same as found from a hydrodynamic model with the same initial density profile and temperature, but without assuming any fast ion losses.

+) For further references see, for example, the second paper of Ref. 4.
II. Hydrodynamic Model with Losses

An amount of energy $E_o$ (Fig. 1) may be transferred within $10^{-9}$ sec into a pure deuterium or deuterium-tritium plasma of solid state density. The plasma will then expand adiabatically from an initial radius $R = R_o$ and an initial temperature $T_o$. This expansion is governed by the law

$$\dot{R} = \left( \frac{5kT_o}{m_o} \left( 1 - (R_o/R)^2 \right) \right)^{1/2}$$  \hspace{1cm} (1)

where $k$ is Boltzmann's constant, $m_o$ is the averaged mass of the plasma particles, and $R = R(t)$ is the actual radius of the sphere at the time $t$. Using Eq. (1) together with the adiabatic law for the temperature $T$, we find for the fusion gain

$$G = \frac{\varepsilon_F}{E_o} \int_0^\infty \int d\mathbf{x}_1 \int d\mathbf{x}_2 \int d\mathbf{x}_3 \frac{m^2(R(t))}{A} \langle \sigma \nu \rangle$$  \hspace{1cm} (2)

where $\varepsilon_F$ is the energy released by one fusion reaction, $A$ is a factor equal to 2 for pure deuterium and equal to 4 for the D-T mixture. To determine the average value $\langle \sigma \nu \rangle$, where $\sigma$ is the reacting cross section and $\nu$ the relative velocity of two ions, we use for $\nu$ a Maxwellian distribution modified in such a way as to take into account that the fast ions leave the plasma without reacting. This modification is found in the following way: The averaged distance $\langle \ell_F \rangle$ which an ion with a velocity $\nu$ travels until it produces a fusion reaction is, because of numerous elastic collisions,

$$\langle \ell_F \rangle = \ell_c \frac{\nu_c}{\nu_F}$$  \hspace{1cm} (3)

where $\nu_F$ is the reaction frequency for fusion processes, $\nu_c$ the ion collision frequency, and $\ell_c$ the mean free path for elastic collisions. If $\langle \ell_F \rangle$ is less than $\bar{R} = \frac{8}{9} R$, which represents an effective radius in the sense of averaging over position and flight direction of the particles, the probability that the particle will react is unity, and if $\bar{R} < \langle \ell_F \rangle$ the pro-
bability is $R / \langle \mathcal{E}_F \rangle$. For fusion gain resulting from Eqs. (1) and (2) we obtain by methods similar to those used before

$$G = \frac{\mathcal{E}_F}{E_0} \nu_0 V_0 \left[ \frac{m_e}{5kT_o} \int_0^{\infty} \frac{dR}{(1-(R/R_o))^{1/2}} \langle \sigma \nu \rangle \right]$$

(4)

with

$$\langle \sigma \nu \rangle = \int_{0}^{\infty} \left[ \min \left( \frac{R}{\langle \mathcal{E}_F (n_e) \rangle} \right) \langle \bar{R} \rangle \right] \sigma(v) \nu^2 \exp \left( -\frac{m_e v^2}{2kT_o} \right) d\nu$$

(5)

where $\bar{m}$ is the reduced mass of the colliding nuclei.

The results of the calculation of $G$ for varying laser energy $E_0$ and different initial volumes $V_0$ at the initial density $n_o = 6 \times 10^{22}$ cm$^{-3}$ of the solid material are shown in Fig. 2. The maximum gain for different values of $V_0$ increases with very good approximation as $E_0^{2/3}$, while the corresponding law for the case without losses is $E_0^{1/3}$. The even-point, $G = 1$, is reached with losses at $E_0 = 3.5 \times 10^7$ joules, while without losses $G$ was $1.6 \times 10^6$ joules. From a set of plots of the kind in Fig. 2 we derive a formula for the maximum $G$ valid for $10^3$ joules $< E_0 < 10^{10}$ joules, and $10^{20} \ cm^{-3} < n_o < 10^{25} \ cm^{-3}$, for a D-T plasma

$$G = \begin{cases} 4.42 \times 10^{-3} \ E_0^{2/3} n_o^{4/3} & \text{if } E_0 n_0 \leq 3.7 \times 10^{15} \text{ joules/cm}^2 \\ 1.47 \times 10^{-19} \ E_0^{1/3} n_o^{2/3} & \text{if } E_0 n_0 \geq 3.7 \times 10^{15} \text{ joules/cm}^2 \end{cases}$$

(6)

For a deuterium plasma we find for the same region of validity

$$G = 3 \times 10^{-3} E_0^{2/3} n_o^{4/3}$$

(7)

At the limits of the range of validity the formulae may derivate from the numerical results by up to 10%.
III. Kinetic Model

It is well known in hydrodynamics from numerical calculations that an initially boxlike density profile becomes Gaussianlike. The same happens if the initial density profile is triangular. As was shown analytically, an initially Gaussian density profile of spatially constant temperature conserves this profile self-similarly. Because of these facts and because it is not completely known, what the real initial state of a plasma produced by laser is, the approximation of the dynamics by a self-similarity expansion of Gaussian density profiles appears to be justified. In the following kinetic treatment, we therefore choose this type of initial conditions.

We use a simplified model in which the dynamics of only one particle species, namely deuterium ions, is taken into account. From the results obtained in this way we refer to what might occur in a D-T plasma. Neglecting the electrons means neglecting space charges. The latter would be true if the pressure of the electrons could be neglected. This, of course, is not the case in real experiments, but the order of magnitude should not be influenced by this. Neglecting the dynamics of the electrons and ion-electron collisions is justified by the small electron mass. We are dealing with a distribution $f(x, v, t)$ for the ions obeying a kinetic equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = \left( \frac{\partial f}{\partial x} \right)_{coll}$$

(8)

The collision term is assumed to be a local one, describing only ion-ion collisions. It is easy to find a solution which makes both sides of Eq. (8) equal to zero. To make the left-hand side vanish, $f$ must depend only on the constants of motion $v$ and $x-ty$. The right-hand side of Eq. (8) postulates that $f$ be an exponential function of the additive collisional invariants $v$ and $v^2$. Such a function is
\[ f = \frac{m_v}{(2\pi k T_0/m_i)^{3/2}} \exp\left[-\frac{1}{k T_0}(x - v t)^2 - \frac{1}{k T_0} \frac{m_i}{2} \nu v^2\right] \]  \hspace{1cm} (9)

where \( T_0, R_0 \) and \( n_0 \) are constants and \( m_i \) is the ion mass.

Equation (9) can be rewritten in the form

\[ f = \frac{m_v}{(2\pi k T_0/m_i)^{3/2}} \left( \frac{\Theta}{T} \right)^{3/2} \exp\left[-\frac{x^2}{R_0^2} \frac{\Theta}{T} \right] \exp\left[\frac{m_i}{2k T_0} \left( \frac{2k \Theta}{m_i R_0^2} \nu \right)^2 \right] \]  \hspace{1cm} (10)

with \( \Theta = T_0 / (1 + 2k T_0 m_v R_0^2 \nu^2) \)

It can be shown that the dynamic expansion is like an adiabatic one with the ratio of the specific heats \( \gamma = 5/3 \).

The local fusion rate is

\[ \frac{1}{2} m^2 \langle \Delta \sigma \rangle = \frac{1}{2} \int d^3 x \int d^3 \nu \delta(\nu - \nu_1) f(\nu_1, \nu, \xi_1, \nu_1, \xi, t) f(\nu_1, \nu, c, \xi_1, \nu_1, c, t) \]  \hspace{1cm} (11)

Introducing new coordinates

\[ \nu = \nu_1 - \nu_2 = \left( \nu_1 - \frac{2k \Theta}{m_i R_0^2} \nu \right) - \left( \nu_2 - \frac{2k \Theta}{m_i R_0^2} \nu \right) \]

\[ \nu = \frac{1}{2} \left( \nu_1 - \frac{2k \Theta}{m_i R_0^2} \nu + \nu_2 - \frac{2k \Theta}{m_i R_0^2} \nu \right) \]

we obtain

\[ \frac{1}{2} m^2 \langle \Delta \sigma \rangle = \frac{m_v^2}{2} \int d^3 x \int d^3 \nu \exp\left[-\frac{m_i t}{2k \Theta} \nu^2\right] \frac{\exp\left[-\frac{m_i t}{2k \Theta} \nu^2\right]}{(2\pi k \Theta/(m_i/2))^{3/2}} \cdot \exp\left[-\frac{2x^2}{R_0^2} \frac{\Theta}{T_0}\right] \exp\left[-\frac{2x^2}{R_0^2} \frac{\Theta}{T_0}\right] \left( \frac{\Theta}{T_0} \right)^{3/2} \]  \hspace{1cm} (12 a)

\[ \frac{1}{2} m^2 \langle \Delta \sigma \rangle = \frac{m_v^2}{2} \left( \frac{\Theta}{T_0} \right)^{3/2} \exp\left[-\frac{2x^2}{R_0^2} \frac{\Theta}{T_0}\right] \int d^3 \xi d^3 \nu \exp\left[-\frac{(m_i/2)^2}{2k \Theta}\right] \exp\left[-\frac{(m_i/2)^2}{2k \Theta}\right] \]  \hspace{1cm} (12 b)

\[ \frac{1}{2} m^2 \langle \Delta \sigma \rangle = \frac{m_v^2}{2} \left( \frac{\Theta}{T_0} \right)^{3} \exp\left[-\frac{2x^2}{R_0^2} \frac{\Theta}{T_0}\right] \langle \Delta \sigma \rangle_0 = \frac{m_i^2 \langle \xi, \nu \rangle}{2} \langle \Delta \sigma \rangle_0 \]  \hspace{1cm} (13)
the expression \( \langle \Delta \nu \rangle \) is identical with the one in Eq. (5) if no attention is paid to losses of fast ions, i.e. the function Min is set equal to unity.

Using a normalization of \( n_o \) such that the total number of ions with a Gaussian profile is the same as these with a box profile, and such that the initial central densities are the same, the ratio of the corresponding fusion gains is

\[
\frac{G_{\text{Box}}}{G_{\text{Gauss}}} = \frac{\int n_o^l d\tau}{\int n_o^{\text{Gauss}} d\tau} = 2^{3/2}
\] (14)

where \( G_{\text{Box}} \) is calculated without allowance for the fast ion losses. This fusion gain \( G_{\text{Box}} \) is therefore by definition the same as discussed earlier 4.

In Fig. 3 we report the results of \( G = G_{\text{Gauss}} \) for a D-T plasma with an initial solid-state density. In Fig. 4 \( G = G_{\text{Gauss}} \) is shown for pure deuterium. Interpolation formulae for the maximum fusion gains for \( 1 \) joule < \( E_o \) < \( 10^{15} \) joule and \( 10^{16} \) cm\(^{-3} \) < \( n_o \) < \( 10^{25} \) cm\(^{-3} \) are

\[
G_{\text{Gauss}} = 3.92 \times 10^{-18} n_o^{1/3} E_o^{4/3} \quad \text{(D-T)}
\]

\[
G_{\text{Gauss}} = 1.79 \times 10^{-19} n_o^{1/4} E_o^{1/3} \quad \text{(D-T)}
\]

Therefore, a D-T plasma will reach the even-point at \( G_{\text{Gauss}} = 4.6 \) MJ and for pure deuterium \( G_{\text{Gauss}} = 4.2 \times 10^4 \) MJ.

IV. Conclusions

We have shown that according to a rather pessimistic estimate laser energies of about \( 3 \times 10^7 \) joules are necessary to reach the even-point, or according to a probably more realistic estimate only \( 4.6 \times 10^6 \) joules. These figures are perhaps not beyond future capabilities, especially as regards chemical lasers.
Acknowledgements

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References


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Fig. 1 Laser energy $E_0$ is transferred to a D-T plasma producing a box-like density profile at given initial values for a following free expansion.

Fig. 2 Fusion gain $G$ with ion losses at varying initial values of the laser energy $E_0$ and volume $V_0$ for an initial density of solid deuterium $n_o = 6 \times 10^{22} \text{ cm}^{-3}$.

Fig. 3 Fusion gains $G = G_{\text{Gauss}}$ for a D-T plasma, for varying laser energy $E_0$ and volume $V_0$ and for an initial density of $n_o = 6 \times 10^{22} \text{ cm}^{-3}$.

Fig. 4 Fusion gains $G = G_{\text{Gauss}}$ for pure deuterium for varying laser energy $E_0$ and volume $V_0$ and for an initial density of $n_o = 6 \times 10^{22} \text{ cm}^{-3}$.
Laser energy $E_0$

Initial values

Temperature $T_0$

Radius $R_0$

Ion density $n_0$

Fig. 1
INITIAL DENSITY \( n_0 = 6 \times 10^{22} \text{ cm}^{-3} \)

D - T

INITIAL VOLUME

\( V_0 = 10^4 \text{ cm}^3 \)

\( 2 \times 10^4 \)

\( 5 \times 10^4 \)

\( 10^3 \)

\( 10^2 \)

\( 10^1 \)

\( 10^0 \)

\( 10^1 \)

\( 10^0 \)

\( 10^{-1} \)

\( 10^{-2} \)

\( 10^{-3} \)

\( 10^{-4} \)

\( 10^{-5} \)

\( 10^{-6} \)

\( 10^{-7} \)

\( 10^{-8} \)

\( 10^{-9} \)

\( 10^{-10} \)

\( 10^4 \)

\( 10^5 \)

\( 10^6 \)

\( 10^7 \)

\( 10^8 \)

\( 10^9 \)

\( 10^{10} \)

FUSION GAIN (FUSION ENERGY/LASER ENERGY)

LASER ENERGY (JOULE)

Fig. 2
INITIAL DENSITY \( n_0 = 6 \times 10^{22} \text{ cm}^{-3} \)

D - D (GAUSSIAN PROFILE)

INITIAL VOLUME

\( V_0 = 5 \times 10^5 \text{ cm}^3 \)

FUSION GAIN (FUSION ENERGY/LASER ENERGY)

LASER ENERGY (JOULE)

Fig. 4