Optimal State Estimation for Cavity Optomechanical Systems

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We demonstrate optimal state estimation for a cavity optomechanical system through Kalman filtering. By taking into account nontrivial experimental noise sources, such as colored laser noise and spurious mechanical modes, we implement a realistic state-space model. This allows us to obtain the conditional system state, i.e., conditioned on previous measurements, with a minimal least-squares estimation error. We apply this method to estimate the mechanical state, as well as optomechanical correlations both in the weak and strong coupling regime. The application of the Kalman filter is an important next step for achieving real-time optimal (classical and quantum) control of cavity optomechanical systems.

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Introduction.—State estimation is a crucial task at the heart of control theory, both in the classical [1] and quantum domain [2]. For Gaussian systems, real-time state estimation can be achieved in an optimal manner using Kalman-Bucy filtering [3,4]. Since many physical systems are approximately Gaussian, Kalman filtering has been successfully implemented for a broad range of uses, for example, for navigation and tracking in aeronautics (including the Apollo project and the Global Positioning System) [5], as well as in the physical sciences, such as for suspension noise cancellation in gravitational wave detection [6], Heisenberg limited atomic magnetometry [7], or quantum-enhanced optical-phase tracking [8–11]. In this Letter, we introduce a new domain of applications by implementing Kalman filtering for cavity optomechanical systems. These systems represent a versatile light-matter interface in which optomechanical interactions inside optical or microwave cavities allow control over optical and mechanical degrees of freedom. While the first investigations go back to the late 1960s in the context of gravitational wave detectors [12,13], it is only the last few years that have seen the development of a completely new generation of micro- and nano-optomechanical solid-state devices with fast-growing application areas from classical sensing to quantum information processing [14].

State estimation of a cavity optomechanical system in real time is key for optimal state control and verification. The outstanding challenge is to obtain reliable information on the mechanical subsystem. In optomechanics, this is done through an optical cavity field, which imposes both additional noise and dynamical backaction effects that have to be taken into account. Until now, reconstructions of the mechanical dynamics have focused either on statistical properties [15,16] or, for real-time reconstructions, on regimes of sufficiently weak coupling and negligible dynamical backaction effects [17–20]. The information obtained from real-time estimation about the mechanical quadratures can be used for active feedback control of the mechanical resonator [19–21]. However, the validity of these reconstruction schemes breaks down when either coupling strength, dynamical backaction effects, or noise become strong. Our Kalman-filtering approach overcomes this limitation and allows us to demonstrate real-time optimal state estimation for cavity optomechanical systems operating in arbitrary parameter regimes. From a quantum physics perspective, the Kalman filter solves the stochastic Schrödinger equation—a stochastic, nonlinear generalization of the Schrödinger equation—which is the canonical way to describe quantum systems subject to a continuous measurement via coupling to electromagnetic fields [2,22–24]. These concepts and their application to mechanical systems have been the subject of extensive theoretical research [25–29], but no experiments in the context of cavity optomechanics have been conducted so far.

Kalman filter.—In quantum theory, just as in classical theories, a continuously observed system can be described by a conditional state [2], i.e., a state that incorporates the total amount of knowledge that an observer has extracted from her set of measurements. Discarding this knowledge yields the unconditional state, which is an incoherent mixture of all possible conditional states. Our goal is to find the (multipartite) conditional states of the full cavity optomechanical system including mechanical and optical subsystems. We restrict ourselves to Gaussian dynamics and measurements, which is a valid assumption for the
Relevant existing realizations of optomechanical systems [14]. For this case, it has been shown [30] that the problem of finding the conditional state can be mapped to a classical estimation problem, which is solved by a Kalman filter. It produces a real-time state estimate from a continuous measurement trajectory, which is optimal in the sense of minimizing the mean-square estimation error. We describe the system by the following (linear) state-space model

\[
\dot{x}_t = A_t x_t + \omega_t, \\
\dot{z}_t = C_t x_t + v_t,
\]

(1a)

(1b)

where \(x\) is a state vector in some appropriately chosen state space (e.g., the phase space of a harmonic oscillator), \(z\) is the outcome of a linear measurement on the system, and \(w\) and \(v\) describe process and measurement noise, respectively. Both \(w\) and \(v\) are assumed to be zero-mean Gaussian white-noise processes, which obey \(\text{Re}(\mathbb{E}[w(t)w^\dagger(t)]) = W_0\delta(t-s)\) and \(\text{Re}(\mathbb{E}[v(t)v^\dagger(t)]) = V_0\delta(t-s)\), where \(\delta\) is the Dirac \(\delta\) function, \(\mathbb{E}[\cdot]\) denotes the expectation value with respect to the initial probability distribution describing system and noise, and \(\text{Re}(\cdot)\) the real part [31]. Process and measurement noise may be correlated, which is described by the cross-correlations \(\text{Re}(\mathbb{E}[w(t)v^\dagger(t)]) = Wv_0\delta(t-s)\) and \(\text{Re}(\mathbb{E}[v(t)w^\dagger(t)]) = Vw_0\delta(t-s)\).

In other words, the conditional Gaussian state \(\hat{x}\_t\) conditioned on a continuous measurement of \(z\), is para-metered by \(x_t\) and \(P_t\) [32]. By averaging over all possible trajectories of \(\hat{x}\_t\), we recover the unconditional state, whose covariance matrix \(\text{Re}(\mathbb{E}[x(t)x^\dagger(t)])\) we can extract from the estimated data by noting that \(\mathbb{E}[((x_t - \hat{x}_t)(x_t - \hat{x}_t)^\dagger)] = 0\), and, thus, \(\text{Re}(\mathbb{E}[x(t)x^\dagger(t)]) = P_t + \mathbb{E}[\hat{x}_t\hat{x}_t^\dagger]\) [2,33].

The Kalman–filter equations (2) describe how the conditional state is iteratively updated [Fig. 1(a)]. First, the estimate \(\hat{x}\) and the covariance \(P_t\) are propagated for an infinitesimal time interval \(dt\) [the first term in Eq. (2a)] and the first three terms in Eq. (2b)] according to the state-space model Eq. (1a). Second, the measurement outcome is incorporated as a Bayesian update that corrects the value of the estimate \(\hat{x}\), and contracts the covariance ellipse [last terms in Eqs. (2a) and (2b)]. The updated values are again propagated by \(dt\), and the procedure is repeated.

**The model.**—We consider a typical cavity optomechanical architecture [Fig. 1(b)], in which a Fabry–Pérot cavity (resonance frequency \(\omega_c\)) coupled to a single mechanical mode [34] is driven by two laser fields (at frequencies \(\omega_{0,d}\), \(\omega_{0,v}\)). The “resonant” beam \((\omega_d = \omega_c)\) acts as a weak probe of the cavity length to stabilize the laser frequency with respect to the cavity resonance; the “detuned” beam \((\omega_d \neq \omega_c)\) induces dynamical backaction effects, e.g., for laser cooling. This captures all relevant scenarios applied in typical optomechanics experiments. The mechanical element has a resonance frequency \(\omega_m\) and energy damping rate \(\gamma_m\). Both cavity modes exhibit decay at a (half width at half maximum) rate \(\kappa = \kappa_1 + \kappa_2\), where \(\kappa_1\) describes the...
input coupler, and $\kappa_2$ accounts for spurious photon losses. The system is described by the (linearized) quantum Langevin equations [35–38]

$$\dot{q} = \alpha_m p, \quad (3a)$$

$$\dot{p} = -\gamma_m q - \gamma_m p + \sum_{i=r,d} g_i (\cos \theta x_i - \sin \theta y_i) + \xi, \quad (3b)$$

$$\dot{x}_i = -\kappa x_i + \Delta_i y_i + g_i \sin \theta q + \sqrt{2\kappa_i} x_i^{in} + \sqrt{2\kappa_i} y_i^{in} + |a_{0,i}| \sin \theta \phi_i, \quad (3c)$$

$$\dot{y}_i = -\kappa y_i - \Delta_i x_i + g_i \cos \theta q + \sqrt{2\kappa_i} y_i^{in} + |a_{0,i}| \cos \theta \phi_i, \quad (3d)$$

where $q$, $p$ ($\langle q, p \rangle = i$) describe the position and momentum of the mirror, and $x_k$, $y_k$ with $\langle x_k, y_k \rangle = i\delta_{lk}$ for $l, k \in \{ r, d \}$, respectively, denote the amplitude and the phase quadrature of the cavity modes of the resonant and detuned beam. The optomechanical coupling to the cavity mode $i$ is given by $g_i = \sqrt{2\gamma_0} |a_{0,i}|$ with $a_{0,i} = \sqrt{2\kappa_i} P_i / \hbar \omega_{0,i} (\kappa + i\Delta_i)$, where $\gamma_0$ is the single-photon coupling strength, $P_i$ is the corresponding driving laser power, and $\Delta_i = \omega_{0,i} - \omega_c$ is the detuning of the respective driving laser (at $a_{0,i}$) with respect to the cavity resonance frequency ($\gamma_0$). The coupling of the mechanics to a thermal bath is modeled by a self-adjoint noise term $\xi$ with $\langle \xi(t) \xi(t') + \xi(t) \xi(t') \rangle = 2\gamma_m (2n + 1) \delta(t - t')$ and $n \approx \kappa T / \hbar \omega_m$ (the mean occupation number of the bath at temperature $T$). Optical shot noise is denoted by $x_i^{in}$, $y_i^{in}$ with variances $\langle x_i^{in}(t) x_i^{in}(s) \rangle = \langle y_i^{in}(t) y_i^{in}(s) \rangle = \frac{1}{2} \delta(t - s)$. Terms proportional to $\delta \phi_i$ and $\phi_i$ describe the classical amplitude and phase noise of the driving lasers [36–38].

Homodyne detection is used to independently measure the generalized quadratures $z_i$ of the reflected optical modes [Fig. 1(b)]. The cavity input-output relations yield

$$z_i' = \langle \sqrt{2\kappa_i} x_i + x_i^{in} + \delta \phi_i \rangle \cos \phi_i + \langle \sqrt{2\kappa_i} y_i + y_i^{in} \rangle \sin \phi_i, \quad (4)$$

where $\delta \phi_i$ describes classical amplitude noise. We model optical losses and inefficient detection as beam-splitter losses parametrized by $\eta$. The measured quantities are rescaled to $z_i = \sqrt{1 - \eta} z_i' + \sqrt{\eta} z_i^{in}$, where $z_i^{in}$ describes additional quantum noise independent of $x_i^{in}$ and $y_i^{in}$, i.e., $\langle x_i^{in}(t) x_i^{in}(s) \rangle = \langle y_i^{in}(t) y_i^{in}(s) \rangle = 0$. Defining the vectors $x_i = [q(t), p(t), x_d(t), y_d(t), x_r(t), y_r(t)]^T$ and $z_i = [z_d(t), z_r(t)]$, Eqs. (3) and (4) can be rewritten in the compact form (1).

Contrary to the idealizing assumptions made above, many of the noise sources in an actual experiment are frequency dependent, here the laser amplitude and phase noise. This needs to be taken into account by properly extending the state-space model. We incorporate three types of laser noise: (i) broadband laser noise originating from the laser itself, (ii) narrow-band Pound-Drever-Hall phase modulation in the resonant beam required for locking the laser to the cavity frequency, and (iii) narrow-band laser noise originating from the feedback loop of the laser lock. Each of these noise sources is experimentally characterized and is modeled independently to match the overall spectral characteristics [39]. Furthermore, we extend the state-space model to incorporate higher-order mechanical modes.

**Measurements and innovations.**—We use the recorded homodyne signals $z_i$ as input to the Kalman filter for estimation of the optomechanical state, which is done offline. Figure 2 shows a 2 $\mu s$ trace of the detector signals (corresponding to 100 sample points), along with the optimal measurement prediction. The prediction shows excellent qualitative agreement with the measured data both in the weak ($g_d < \kappa$) and strong coupling regime ($g_d > \kappa$). Quantitatively, the validity of the estimation is assessed by the innovation sequence $\nu_i = z_i - C \hat{x}_i$, i.e.,

![FIG. 2 (color online). Measurement signals and Kalman-filter predictions. Shown are the measurement signals of the two homodyne detections of the detuned and resonant beam, $z_d(t)$ (red dots) and $z_r(t)$ (blue dots), respectively, and their Kalman-filter predictions $\hat{z}_d(t)$ (gray line) both for the weak (left) and the strong coupling regime (right). Error bars of the prediction ($\pm 2\sigma$) are indicated by the width of the gray line. Kalman-filter innovations $\nu_d(t) = z_d(t) - \hat{z}_d(t)$ are plotted below each data set and demonstrate the accuracy of the implemented filter. To assess the performance of the filter, we calculate the fraction of normalized innovations that are contained in a two-sided 95% confidence region ($\pm 2\sigma$ indicated by the lines) of a zero-mean Gaussian distribution (beside each plot). The experiment was performed at room temperature with a micromechanical oscillator of $\omega_m = 2\pi \times 1.278$ MHz, $\gamma_m = 2\pi \times 265$ Hz, and optomechanical parameters $\kappa = 0.34\omega_m$, $g_0 = 2\pi \times 7.7$ Hz [39]. For both coupling strengths of the detuned beam ($\Delta_d = \omega_m$), we use $\eta_d \approx 0$, $\Delta_r = 0$, $g_r = 0.2\kappa$, and $\phi_r = \pi/2$. Note that the fast oscillation of $z_d(t)$, which is due to the 20 MHz Pound-Drever-Hall phase modulation for frequency locking, is taken into account by the Kalman filter.
the difference between the predicted measurement $\hat{z}_r = C_r \hat{x}_r$ of the Kalman filter and the actual measurement outcome $z_r$. For an optimally working filter, $\nu_r$ must be a Gaussian zero-mean white-noise process with a variance given by $\mathbb{E}[\nu_r \nu_r^\dagger] = C_r P_{z_q} C_r^\dagger + V$. We use this fact to fine-tune model parameters starting from their independently determined values. The statistics of $\nu_r$ of the resulting Kalman filter closely matches these criteria, hence, demonstrating the accuracy of the filter (Fig. 2; see, also, Ref. [39] for further statistical analysis).

**Estimation of optomechanical quadratures.**—Kalman filtering provides direct, real-time access both to the optical intracavity quadratures and to the mechanical degree of freedom in a cavity optomechanical system [Fig. 3(a)]. In the weak coupling regime, the thermally driven mechanical motion and its coupling to the optical intracavity fields is visible. Clearly, the mechanical motion modulates both quadratures $x_d, y_d$ of the detuned beam [14], which couples to the mirror via the optomechanical beam-splitter interaction. The situation is different for the resonant beam, whose amplitude quadrature $x_r$ contains shot noise only, while its phase quadrature $y_r$, couples to the mechanical position.

The phase-space representation captures the essence of Kalman filtering. The estimated mechanical quadratures rotate in phase space [Fig. 3(b)]. Their probability distribution along each mechanical quadrature is shown as histograms beside each axis and demonstrates the Gaussian nature of the micromirror motion. We compare the uncertainty ellipse of the unconditional (dashed line) and conditional (solid line) mechanical state, i.e., the area in which we expect with 95% probability to find the mechanical quadratures. For a purely thermal state, the area of the unconditional ellipse is proportional to the thermal occupation number $\bar{n}$. In the weak coupling regime, the information provided by the measurement update leads to a clear reduction in the uncertainty (a factor of 27 in effective temperature), which is the optimal one for the given coupling strength [52]. In the strong coupling regime, laser cooling has already significantly diminished the thermally induced uncertainty of the unconditional state. In addition, the cavity dynamics introduces a notable ellipticity in the phase-space distribution [53]. The conditional state uncertainty is similar to the weak coupling situation. This is because for technical reasons the signal power at the homodyne detectors was kept constant for both coupling strengths, which means that the stronger detuned optical drive beam does not provide more information on the system state.

Figure 3(c) shows real-time estimates of the optomechanical correlations between mechanical position and the phase quadrature of the resonant beam. Analogous to the mechanical phase space, the conditional state uncertainties are strongly reduced, reflecting the real-time information gain on the optomechanical correlations.

**Conclusion.**—We have successfully implemented Kalman filtering for optimal state estimation of cavity optomechanical systems. Its accuracy crucially relies on an accurate state-space model of the specific experiment. The applications of this method in the domain of optomechanics are manifold. For example, Kalman filtering enables mechanical feedback control in the quantum regime. While in this work we operate the filter off-line, its real-time application in the frequency range investigated here is feasible using current field programmable gate array hardware [39]. The optimality of the filter guarantees that the reduction in conditional state uncertainty corresponds to the maximal cooling one can achieve through active feedback at this specific coupling strength. As a consequence, ground-state cooling is readily achievable by combining Kalman filtering with measurements in the strong cooperativity regime [33]. This regime has been reached in current experiments [21,54–56]. In our case, it requires cryogenic
cooling of the mechanical environment to 300 mK and quality factors above 10^10. As another example, mechanical sensing requires precise knowledge of the system dynamics in the absence of the external impetus, which is equivalent to the task of implementing the optimal estimator, i.e., the Kalman filter. The same is true for the task of characterizing or reconstructing an optomechanical quantum state (for example, in terms of entanglement), where the relevant information is often encoded in the covariance matrix \( P_t \).

One fascinating prospect there is the generation of entanglement of macroscopic test masses through measurement [29, 57]. In summary, Kalman filtering adds a significant performance advantage for classical and quantum control of cavity optomechanical systems.

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[31] Taking the real part of the covariance matrices that describes the noise processes is only necessary for quantum processes due to their noncommutative nature. Although it is not necessary for classical systems, we choose this explicitly real form for the sake of a consistent presentation.

[32] One can also adopt a quantum-optical interpretation of \( \hat{x}_t \). Formally integrating (and assuming vanishing initial conditions) gives \( \hat{x}_t = \int_{-\infty}^{\infty} K(t,s)z_s ds \) with an integral kernel \( K \) depending on \( A_t \), \( C_t \), and \( K_t \). Thus, \( \hat{x}_t \) is formally equivalent to a (unnormalized) bosonic mode extracted from the output process \( z_t \). In a quantum-optical setting, this could be, for example, a temporal light mode extracted from the output light of a cavity.


[34] The generalization to several mechanical modes is straightforward, and, in fact, it is included in the full model of our system.


[39] See the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.114.223601, for details on the state space model implementation of our experiment, the statistical analysis of the innovation sequence and the feasibility of real-time feedback. It contains Refs. [37,40–51].


[52] Recall that the optimality of the Kalman filter ensures that the conditional state uncertainty is minimized for a given coupling strength.


