ANCIENT SOLUTIONS TO THE RICCI FLOW WITH PINCHED CURVATURE

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Abstract
We show that any ancient solution to the Ricci flow which satisfies a suitable curvature pinching condition must have constant sectional curvature.

1. Introduction
In this article, we study ancient solutions to the Ricci flow on compact manifolds. Recall that a one-parameter family of metrics $g(t)$ on a compact manifold $M$ evolves by the Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2 \text{Ric}_{g(t)}.$$ 

A solution to the Ricci flow is called ancient if it is defined on a time interval $(-\infty, T)$. Ancient solutions typically arise in the study of singularities to the Ricci flow (see, e.g., [15], [16], [19], [20]).

P. Daskalopoulos, R. Hamilton, and N. Šešum [11] have recently obtained a complete classification of all ancient solutions to the Ricci flow in dimension 2 (see also [10], where the analogous question for the curve shortening flow is studied). V. Fateev [12] has constructed an interesting example of an ancient solution in dimension 3. L. Ni [18] showed that any ancient solution to the Ricci flow which is of type I, is $\kappa$-noncollapsed, and has positive curvature operator has constant sectional curvature.

In this article, we show that any ancient solution to the Ricci flow in dimension $n \geq 3$ which satisfies a suitable curvature pinching condition must have constant sectional curvature. In dimension 3, we impose a condition on the Ricci tensor of $(M, g(t))$.

THEOREM 1
Let $M$ be a compact three-manifold, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on $M$. Moreover, suppose that there exists a uniform constant $\rho > 0$

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such that

\[ \text{Ric}_{g(t)} \geq \rho \text{scal}_{g(t)} g(t) \geq 0 \]

for all \( t \in (-\infty, 0) \). Then the manifold \( (M, g(t)) \) has constant sectional curvature for each \( t \in (-\infty, 0) \).

Fateev’s example shows that the pinching condition for the Ricci tensor cannot be removed. The proof of Theorem 1 relies on a new interior estimate for the Ricci flow in dimension 3 (see Proposition 3 below). The proof of this estimate relies on the maximum principle and will be presented in Section 2.

In dimension \( n \geq 4 \), we prove the following result.

**Theorem 2**

Let \( M \) be a compact manifold of dimension \( n \geq 4 \), and let \( g(t), t \in (-\infty, 0) \), be an ancient solution to the Ricci flow on \( M \). Moreover, suppose that there exists a uniform constant \( \rho > 0 \) with the following property: for each \( t \in (-\infty, 0) \), the curvature tensor of \( (M, g(t)) \) satisfies

\[
R_{g(t)}(e_1, e_3, e_1, e_3) + \lambda^2 R_{g(t)}(e_1, e_4, e_1, e_4) \\
+ R_{g(t)}(e_2, e_3, e_2, e_3) + \lambda^2 R_{g(t)}(e_2, e_4, e_2, e_4) \\
- 2\lambda R_{g(t)}(e_1, e_2, e_3, e_4) \geq \rho \text{scal}_{g(t)} \geq 0
\]

for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \) and all \( \lambda \in [0, 1] \). Then the manifold \( (M, g(t)) \) has constant sectional curvature for each \( t \in (-\infty, 0) \).

Theorem 2 again follows from pointwise curvature estimates which are established using the maximum principle (see Corollary 7 below). In dimension \( n \geq 4 \), the evolution equation for the curvature tensor is much more complicated, and our estimates are not as explicit as in the three-dimensional case. In order to handle the higher-dimensional case, we use the invariant curvature conditions introduced in [3] and [7]. These ideas also play a key role in the proof of the differentiable sphere theorem (see [7], [8]).

**2. Proof of Theorem 1**

**Proposition 3**

Let \( M \) be a compact three-manifold, and let \( g(t), t \in [0, T) \), be a solution to the Ricci flow on \( M \). Moreover, suppose that there exists a uniform constant \( \rho \in (0, 1) \) such
that

$$\operatorname{Ric}_{g(t)} \geq \rho \operatorname{scal}_{g(t)} g(t) \geq 0$$

for each $t \in [0, T)$. Then, for each $t \in (0, T)$, the curvature tensor of $(M, g(t))$ satisfies the pointwise estimate

$$|\mathcal{Ric}_{g(t)}| \leq \left(\frac{3}{2t}\right)^{\sigma} \operatorname{scal}_{g(t)}^{2-\sigma},$$

where $\sigma = \rho^2$.

**Proof**

The assertion is trivial if $(M, g(0))$ is Ricci flat. Hence, it suffices to consider the case that $(M, g(0))$ is not Ricci flat. By the maximum principle, the manifold $(M, g(t))$ has strictly positive scalar curvature for all $t \in (0, T)$.

We next define a function $f : M \times (0, T) \to \mathbb{R}$ by

$$f = \operatorname{scal}^{\sigma-2} |\mathcal{Ric}|^{\sigma},$$

where $\sigma = \rho^2$. It is easy to see that $f \leq \operatorname{scal}^{\sigma}$. Moreover, it follows from [13, Lemma 10.5] that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1 - \sigma)}{\operatorname{scal}} \partial_k \operatorname{scal} \partial^k f + 2 \operatorname{scal}^{\sigma-3} [\sigma |\mathcal{Ric}|^{\sigma} |\mathcal{Ric}|^{2} - 2P],$$

where $P$ is a polynomial expression in the eigenvalues of the Ricci tensor. By assumption, we have $\mathcal{Ric} \geq \rho \operatorname{scal} g$. Hence, it follows from [13, Lemma 10.7] that

$$P \geq \rho^2 |\mathcal{Ric}|^{2} |\mathcal{Ric}|^{\sigma}. $$

This implies that

$$2P - \sigma |\mathcal{Ric}|^{\sigma} |\mathcal{Ric}|^{\sigma} \geq \sigma |\mathcal{Ric}|^{2} |\mathcal{Ric}|^{\sigma}$$

$$\geq \frac{1}{3} \sigma \operatorname{scal}^{2} |\mathcal{Ric}|^{\sigma}$$

$$= \frac{1}{3} \sigma \operatorname{scal}^{4-\sigma} f$$

$$\geq \frac{1}{3} \sigma \operatorname{scal}^{3-\sigma} f^{1+1/\sigma}.$$

Putting these facts together, we conclude that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1 - \sigma)}{\operatorname{scal}} \partial_k \operatorname{scal} \partial^k f - \frac{2}{3} \sigma f^{1+1/\sigma}.$$
Using the maximum principle, we obtain
\[ f \leq \left( \frac{3}{2t} \right)^{\sigma}. \]
This completes the proof.

COROLLARY 4

Let \( M \) be a compact three-manifold, and let \( g(t) \), \( t \in (-\infty, 0) \), be an ancient solution to the Ricci flow on \( M \). Moreover, suppose that there exists a uniform constant \( \rho \in (0, 1) \) such that
\[ \text{Ric}_{g(t)} \geq \rho \text{ scal}_{g(t)} g(t) \geq 0 \]
for each \( t \in (-\infty, 0) \). Then the manifold \((M, g(t))\) has constant sectional curvature for each \( t \in (-\infty, 0) \).

Proof

It follows from Proposition 3 that \( |\text{Ric}_{g(t)}|^2 = 0 \) for each \( t \in (-\infty, 0) \). Therefore, the manifold \((M, g(t))\) has constant sectional curvature for each \( t \in (-\infty, 0) \).

3. The higher-dimensional case

In this section, we develop some general tools that will be used in the proof of Theorem 2. To that end, we fix an integer \( n \geq 4 \). Moreover, we denote by \( \mathcal{C}_B(\mathbb{R}^n) \) the space of algebraic curvature tensors on \( \mathbb{R}^n \). Given any algebraic curvature tensor \( R \in \mathcal{C}_B(\mathbb{R}^n) \), we define an algebraic curvature tensor \( Q(R) \in \mathcal{C}_B(\mathbb{R}^n) \) by
\[
Q(R)_{ijkl} = \sum_{p,q=1}^{n} R_{ijpq} R_{klpq} + 2 \sum_{p,q=1}^{n} (R_{ipkq} R_{jlpq} - R_{iplq} R_{jpkq}).
\]

The expression \( Q(R) \) arises naturally in the evolution equation for the curvature tensor under Ricci flow (see [14]; see also [5, Section 2.3]). The ordinary differential equation (ODE) \( \frac{d}{dt} R = Q(R) \) on the space \( \mathcal{C}_B(\mathbb{R}^n) \) will be referred to as the Hamilton ODE.

We next consider a cone \( C \subset \mathcal{C}_B(\mathbb{R}^n) \). We say that the cone \( C \) has property (\( * \)) if the following conditions are met:

(i) \( C \) is closed, convex, and \( O(n) \)-invariant.

(ii) \( C \) is transversally invariant under the Hamilton ODE \( \frac{d}{dt} R = Q(R) \).

(iii) Every algebraic curvature tensor \( R \in C \setminus \{0\} \) has positive scalar curvature.

(iv) The curvature tensor \( I_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \) lies in the interior of \( C \).

In the remainder of this section, we assume that \( C \subset \mathcal{C}_B(\mathbb{R}^n) \) is a cone satisfying (\( * \)). Then \( Q(R) \) lies in the interior of the tangent cone \( T_R C \) for all \( R \in C \setminus \{0\} \). By
continuity, we can find a real number \( \alpha_0 > 0 \) such that
\[
Q(R + \alpha \text{scal}(R) I) - \alpha_0^2 \text{scal}(R)^2 I \in T_R C
\]
for all \( R \in C \setminus \{0\} \) and all \( \alpha \in [0, \alpha_0] \). Moreover, there exists a real number \( \Lambda > 0 \) such that \( |\text{Ric}(R)| \leq \Lambda \text{scal}(R) \) for all \( R \in C \). Let
\[
\delta = \min\left\{ \frac{1}{2n(n-1)}, \frac{\alpha_0}{2}, \frac{\alpha_0^2}{4(1 + 2\Lambda^2)} \right\} > 0.
\]
For each \( t \in [0, \delta] \), we define a subset \( F(t) \subset C_B(\mathbb{R}^n) \) by
\[
F(t) = \{ R \in C : R + (1 - t \text{scal}(R)) I \in C \}.
\]
Clearly, \( F(t) \) is closed, convex, and \( O(n) \)-invariant. Moreover, \( F(0) = C \).

**Lemma 5**
Suppose that \( R \) is an algebraic curvature tensor on \( \mathbb{R}^n \) with the property that \( R \in C \) and \( R + (1 - t \text{scal}(R)) I \in C \) for some \( t \in [0, \delta] \). Then
\[
Q(R) - \text{scal}(R) I - 2t |\text{Ric}(R)|^2 I
\]
lies in the interior of the tangent cone to \( C \) at the point \( R + (1 - t \text{scal}(R)) I \).

**Proof**
If \( t \text{scal}(R) < 1 \), then the sum \( R + (1 - t \text{scal}(R)) I \) lies in the interior of \( C \). In this case, the assertion is trivial.

Hence, it suffices to consider the case \( t \text{scal}(R) \geq 1 \). For abbreviation, let
\[
S = R + (1 - t \text{scal}(R)) I \in C.
\]
Since \( t \in [0, \delta] \), we have
\[
\text{scal}(S) > (1 - n(n-1)t) \text{scal}(R) \geq \frac{1}{2} \text{scal}(R).
\]
Hence, if we put
\[
\alpha = \frac{t \text{scal}(R) - 1}{\text{scal}(S)},
\]
then we have \( 0 \leq \alpha < 2t \leq \alpha_0 \). Since \( S \in C \setminus \{0\} \), it follows that
\[
Q(S + \alpha \text{scal}(S) I) - \alpha_0^2 \text{scal}(S)^2 I \in T_S C
\]
by the definition of $\alpha_0$. We next observe that
\[ S + \alpha \operatorname{scal}(S) I = R \]
and
\[ \alpha_0^2 \operatorname{scal}(S)^2 > \frac{\alpha_0^2}{4} \operatorname{scal}(R)^2 \geq (1 + 2 \Lambda^2) \alpha_0^2 \operatorname{scal}(R)^2 \geq \operatorname{scal}(R) + 2t |\operatorname{Ric}(R)|^2. \]
Putting these facts together, we conclude that
\[ Q(R) - \operatorname{scal}(R) I - 2t |\operatorname{Ric}(R)|^2 I \]
lies in the interior of the tangent cone $T_SC$. This completes the proof. \qed

**PROPOSITION 6**

Suppose that $R(t)$ is a solution of the Hamilton ODE $\frac{d}{dt} R(t) = Q(R(t))$ which is defined on some time interval $[t_0, t_1] \subset [0, \delta]$. If $R(t_0) \in F(t_0)$, then $R(t) \in F(t)$ for all $t \in [t_0, t_1]$.

**Proof**

By assumption, we have $R(t_0) \in C$. Since $C$ is invariant under the Hamilton ODE, we conclude that $R(t) \in C$ for all $t \in [t_0, t_1]$. Hence, it suffices to show that $R(t) + (1 - t \operatorname{scal}(R(t))) I \in C$ for all $t \in [t_0, t_1]$.

For abbreviation, let
\[ S(t) = R(t) + (1 - t \operatorname{scal}(R(t))) I \]
for all $t \in [t_0, t_1]$. Since $R(t)$ is a solution of the Hamilton ODE, we have
\[ \frac{d}{dt} S(t) = Q(R(t)) - \operatorname{scal}(R(t)) I - 2t |\operatorname{Ric}(R(t))|^2 I \]
for all $t \in [t_0, t_1]$. We claim that $S(t) \in C$ for all $t \in [t_0, t_1]$. Suppose this false. We define a real number $\tau$ by
\[ \tau = \inf \{ t \in [t_0, t_1] : S(t) \notin C \}. \]

By the definition of $\tau$, we have $\tau \in [0, \delta]$ and $S(\tau) \in C$. Furthermore, we have $R(\tau) \in C$. Hence, Lemma 5 implies that the derivative $\frac{d}{dt} S(t)|_{t=\tau}$ lies in the interior of the tangent cone $T_{S(t)}C$. By [5, Proposition 5.4], there exists a real number $\varepsilon > 0$ such that $S(t) \in C$ for all $t \in [\tau, \tau + \varepsilon]$. This contradicts the definition of $\tau$. \qed
COROLLARY 7
Let $\delta$ be defined as above. Moreover, let $g(t)$, $t \in [0, \delta]$, be a solution to the Ricci flow on a compact $n$-dimensional manifold $M$. Finally, we assume that the curvature tensor of $(M, g(0))$ lies in the cone $C$ for all points $p \in M$. Then
\[ R_{g(t)} + (1 - t \text{ scal}_{g(t)}) I \in C \]
for all points $(p, t) \in M \times [0, \delta]$.

Proof
By assumption, the curvature tensor of $(M, g(0))$ lies in the set $F(0)$ for all points $p \in M$. Using Proposition 6 and the maximum principle (see [9, Theorem 3]), we conclude that the curvature tensor of $(M, g(t))$ lies in the set $F(t)$ for all points $(p, t) \in M \times [0, \delta]$. This proves the assertion.

COROLLARY 8
Let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact $n$-dimensional manifold $M$. Moreover, suppose that the curvature tensor of $(M, g(t))$ lies in the cone $C$ for all $t \in (-\infty, 0)$. Then
\[ R_{g(t)} - \delta \text{ scal}_{g(t)} I \in C \]
for all points $(p, t) \in M \times (-\infty, 0)$.

Proof
Fix a time $\tau \in (-\infty, 0)$, and fix a real number $\sigma > 0$. We define a one-parameter family of metrics $\tilde{g}(t)$, $t \in [0, \delta]$, by
\[ \tilde{g}(t) = \sigma g\left(\frac{t - \delta}{\sigma} + \tau\right). \]
Clearly, the metrics $\tilde{g}(t)$, $t \in [0, \delta]$, form a solution to the Ricci flow. By assumption, the curvature tensor of $(M, \tilde{g}(0))$ lies in the cone $C$ for all points $p \in M$. Hence, it follows from Corollary 7 that
\[ R_{\tilde{g}(\delta)} + (1 - \delta \text{ scal}_{\tilde{g}(\delta)}) I \in C \]
for all points $p \in M$. This implies that
\[ R_{g(t)} + (\sigma - \delta \text{ scal}_{g(t)}) I \in C \]
for all points $p \in M$. Taking the limit as $\sigma \to 0$, we conclude that
\[ R_{g(t)} - \delta \text{ scal}_{g(t)} I \in C \]
for all points $p \in M$. Since $\tau \in (-\infty, 0)$ is arbitrary, the assertion follows.
THEOREM 9
Let $C(s)$, $s \in [0, 1]$, be a family of cones in $C_B(\mathbb{R}^n)$ satisfying property $(\ast)$. Moreover, suppose that the cones $C(s)$ vary continuously in $s$. Finally, let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact $n$-dimensional manifold $M$ such that $R_{g(t)} \in C(0)$ for all points $(p, t) \in M \times (-\infty, 0)$. Then $R_{g(t)} \in C(1)$ for all $(p, t) \in M \times (-\infty, 0)$.

Proof
Let $\mathcal{S}$ denote the set of all real numbers $s \in [0, 1]$ with the property that $R_{g(t)} \in C(s)$ for all points $(p, t) \in M \times (-\infty, 0)$. We claim that $\mathcal{S} = [0, 1]$.

Clearly, $\mathcal{S}$ is closed and nonempty. We next show that $\mathcal{S}$ is an open subset of $[0, 1]$. To that end, we fix a real number $s_0 \in \mathcal{S}$. Then $R_{g(t)} \in C(s_0)$ for all points $(p, t) \in M \times (-\infty, 0)$. By Corollary 8, there exists a real number $\delta > 0$ such that $R_{g(t)} - \delta \text{scal}_{g(t)} I \in C(s_0)$ for all points $(p, t) \in M \times (-\infty, 0)$. Since the cones $C(s)$ vary continuously in $s$, there exists a real number $\varepsilon > 0$ such that $R_{g(t)} \in C(s)$ for all points $(p, t) \in M \times (-\infty, 0)$ and all $s \in [s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1]$. Consequently, we have $[s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1] \subset \mathcal{S}$. This shows that $\mathcal{S}$ is an open subset of $[0, 1]$. Thus, we conclude that $\mathcal{S} = [0, 1]$, as claimed.

4. Proof of Theorem 2
We now describe the proof of Theorem 2. As in the previous section, we fix an integer $n \geq 4$. We denote by $\tilde{C}$ and $\hat{C}$ the cones introduced in [3] and [7]. The cone $\tilde{C}$ consists of all algebraic curvature tensors $R \in C_B(\mathbb{R}^n)$ satisfying

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) - 2\lambda R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0, 1]$. Similarly, the cone $\hat{C}$ consists of all algebraic curvature tensors $R \in C_B(\mathbb{R}^n)$ satisfying

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda \mu R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Similarly, the cone $\hat{C}$ consists of all algebraic curvature tensors $R \in C_B(\mathbb{R}^n)$ satisfying

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda \mu R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0, 1]$. Similarly, the cone $\hat{C}$ consists of all algebraic curvature tensors $R \in C_B(\mathbb{R}^n)$ satisfying

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) - 2\lambda \mu R(e_1, e_2, e_3, e_4) \geq 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$. Consequently, we have $[s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1] \subset \mathcal{S}$. This shows that $\mathcal{S}$ is an open subset of $[0, 1]$. Thus, we conclude that $\mathcal{S} = [0, 1]$, as claimed.
for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n \) and all \( \lambda, \mu \in [0, 1] \). The cones \( \hat{C} \) and \( \check{C} \) are both invariant under the Hamilton ODE \( \frac{d}{dt} R = Q(R) \). A detailed discussion of these cones can be found in [5, Chapter 7].

We next describe a family of invariant curvature cones interpolating between the cone \( \check{C} \) and the cone \( \hat{C} \). For each \( s \in (0, \infty) \), we denote by \( \check{C}(s) \) the set of all algebraic curvature tensors \( R \in \mathcal{C}_B(\mathbb{R}^n) \) such that

\[
R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\
+ \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) \\
- 2\lambda \mu R(e_1, e_2, e_3, e_4) + \frac{1}{s} (1 - \lambda^2) (1 - \mu^2) \text{scal}(R) \geq 0
\]

for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n \) and all \( \lambda, \mu \in [0, 1] \). Clearly, \( \check{C}(s) \) is a closed, convex cone, which is invariant under the natural action of \( O(n) \). Moreover, we have \( \hat{C} \subset \check{C}(s) \subset \check{C} \) for each \( s \in (0, \infty) \). The following result is an immediate consequence of [3, Proposition 10].

**Proposition 10**

For each \( s \in (0, \infty) \), the cone \( \check{C}(s) \) is invariant under the Hamilton ODE \( \frac{d}{dt} R = Q(R) \).

**Proof**

Let us fix a real number \( s \in (0, \infty) \). Moreover, let \( R(t), t \in [0, T) \), be a solution of the Hamilton ODE such that \( R(0) \in \check{C}(s) \). We claim that \( R(t) \in \check{C}(s) \) for all \( t \in [0, T) \). Without loss of generality, we may assume that \( \text{scal}(R(0)) = s \). This implies that

\[
R(0)(e_1, e_3, e_1, e_3) + \lambda^2 R(0)(e_1, e_4, e_1, e_4) \\
+ \mu^2 R(0)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(0)(e_2, e_4, e_2, e_4) \\
- 2\lambda \mu R(0)(e_1, e_2, e_3, e_4) + \frac{1}{s} (1 - \lambda^2) (1 - \mu^2) \geq 0
\]

for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n \) and all \( \lambda, \mu \in [0, 1] \). Hence, [3, Proposition 10] implies that

\[
R(t)(e_1, e_3, e_1, e_3) + \lambda^2 R(t)(e_1, e_4, e_1, e_4) \\
+ \mu^2 R(t)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(t)(e_2, e_4, e_2, e_4) \\
- 2\lambda \mu R(t)(e_1, e_2, e_3, e_4) + (1 - \lambda^2) (1 - \mu^2) \geq 0
\]
for all orthonormal four-frames \( \{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n \), all \( \lambda, \mu \in [0, 1] \), and all \( t \in [0, T] \). Since \( \text{scal}(R(t)) \geq \text{scal}(R(0)) = s \), we conclude that \( R(t) \in \tilde{C}(s) \) for all \( t \in [0, T] \). \( \square \)

After these preparations, we now present the proof of Theorem 2.

**Theorem 11**
Assume that \( g(t), t \in (-\infty, 0) \), is an ancient solution to the Ricci flow on a compact \( n \)-dimensional manifold \( M \). Moreover, we assume that there exists a uniform constant \( \rho > 0 \) such that

\[
R_g(t) - \rho \text{scal}_g(t) I \in \tilde{C}
\]

for all points \( (p, t) \in M \times (-\infty, 0) \). Then the manifold \( (M, g(t)) \) has constant sectional curvature for each \( t \in (-\infty, 0) \).

**Proof**
Consider the one-parameter family of cones \( \hat{C}(s), s \in (0, \infty) \), defined in [7]. It is shown in [7] that the cone \( \hat{C}(s) \) has property \((*)\) for each \( s \in (0, \infty) \). Furthermore, the cones \( \hat{C}(s) \) vary continuously in \( s \).

By assumption, there exists a real number \( s_0 \in (0, \infty) \) such that \( R_g(t) \in \hat{C}(s_0) \) for all points \( (p, t) \in M \times (-\infty, 0) \). Using Theorem 9, we conclude that \( R_g(t) \in \hat{C}(s) \) for all points \( (p, t) \in M \times (-\infty, 0) \) and all \( s \in (0, \infty) \). Consequently, the manifold \( (M, g(t)) \) has constant sectional curvature for each \( t \in (-\infty, 0) \). This completes the proof of Theorem 11. \( \square \)

**Theorem 12**
Assume that \( g(t), t \in (-\infty, 0) \), is an ancient solution to the Ricci flow on a compact \( n \)-dimensional manifold \( M \). Moreover, we assume that there exists a uniform constant \( \rho > 0 \) such that

\[
R_g(t) - \rho \text{scal}_g(t) I \in \tilde{C}
\]

for all points \( (p, t) \in M \times (-\infty, 0) \). Then the manifold \( (M, g(t)) \) has constant sectional curvature for each \( t \in (-\infty, 0) \).

**Proof**
By assumption, we have

\[
R_g(t) - \rho \text{scal}_g(t) I \in \tilde{C}
\]
for all points \((p, t) \in M \times (-\infty, 0)\). Hence, we can find a real number \(s_0 \in (0, \infty)\) such that

\[
R_{g(t)} - \frac{1}{2} \rho \text{scal}_{g(t)} I \in \tilde{C}(s_0)
\]

for all points \((p, t) \in M \times (-\infty, 0)\).

We next consider a pair of real numbers \(a, b\) such that \(2a = 2b + (n - 2)b^2\) and \(b \in \left(0, \frac{\sqrt{2n(n-2)}}{n(n-2)}\right]\). Following [2], we define a linear transformation \(\ell_{a,b} : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)\) by

\[
\ell_{a,b}(R) = R + b \text{Ric}(R) \otimes \text{id} + \frac{1}{n} (a - b) \text{scal}(R) \text{id} \otimes \text{id},
\]

where \(\otimes\) denotes the Kulkarni-Nomizu product (see, e.g., [1, Definition 1.110]). If we choose \(b \in \left(0, \frac{\sqrt{2n(n-2)}}{n(n-2)}\right]\) sufficiently small, then

\[
R_{g(t)} \in \ell_{a,b}(\tilde{C}(s_0))
\]

for all points \((p, t) \in M \times (-\infty, 0)\).

By Proposition 10, the cone \(\tilde{C}(s)\) is invariant under the Hamilton ODE for each \(s \in (0, \infty)\). Consequently, the cone \(\ell_{a,b}(\tilde{C}(s))\) is transversally invariant under the Hamilton ODE for each \(s \in (0, \infty)\) (see [2, Proposition 3.2]). Therefore, the cone \(\ell_{a,b}(\tilde{C}(s))\) has property (\(*\)) for each \(s \in (0, \infty)\). Moreover, if we fix \(a\) and \(b\), then the cones \(\ell_{a,b}(\tilde{C}(s))\) vary continuously in \(s\). Using Theorem 9, we conclude that \(R_{g(t)} \in \ell_{a,b}(\tilde{C}(s))\) for all points \((p, t) \in M \times (-\infty, 0)\) and all \(s \in (0, \infty)\). Taking the limit as \(s \to \infty\), we obtain \(R_{g(t)} \in \ell_{a,b}(\hat{C})\) for all points \((p, t) \in M \times (-\infty, 0)\). Hence, it follows from Theorem 11 that \((M, g(t))\) has constant sectional curvature for each \(t \in (-\infty, 0)\). \(\square\)

5. Ancient solutions satisfying a diameter bound

In this final section, we study ancient solutions to the Ricci flow satisfying a suitable diameter bound. Throughout this section, we assume that \(M\) is a compact manifold of dimension \(n\) and that \(g(t), t \in (-\infty, 0)\), is a solution to the Ricci flow on \(M\). The following proposition is a consequence of the differential Harnack inequality established in [4].

**Lemma 13**

Suppose that the curvature tensor of \((M, g(t))\) lies in the cone \(\hat{C}\) for each \(t \in (-\infty, 0)\). Then

\[
\inf_M \text{scal}_{g(\tau/2)} \geq \exp\left(-\frac{\text{diam}(M, g(\tau))^2}{|\tau|}\right) \sup_M \text{scal}_{g(\tau)}
\]

for all \(\tau \in (-\infty, 0)\).
Proof
Fix an arbitrary pair of points $p, q \in M$. We can find a smooth path $\gamma : [\tau, \tau/2] \to M$ such that $\gamma(\tau) = p$, $\gamma(\tau/2) = q$, and

$$|\gamma'(t)|_{g(t)} = \frac{2d_{g(t)}(p, q)}{|\tau|}.$$ 

This implies that

$$|\gamma'(t)|_{g(t)} \leq \frac{2d_{g(t)}(p, q)}{|\tau|}$$

for all $t \in [\tau, \tau/2]$. Using the trace Harnack inequality (see [4, Proposition 13]), we obtain that

$$\frac{\partial}{\partial t} \text{scal} + 2 \partial_i \text{scal} v^i \geq -2 \text{Ric}(v, v)$$

for every tangent vector $v$. Putting $v = \gamma'(t)/2$ gives

$$\frac{d}{dt} \text{scal}_{g(t)}(\gamma(t)) \geq -\frac{1}{2} \text{Ric}_{g(t)}(\gamma'(t), \gamma'(t))$$

$$\geq -\frac{1}{2} \text{scal}_{g(t)}(\gamma(t)) |\gamma'(t)|_{g(t)}^2$$

$$\geq -\frac{2d_{g(t)}(p, q)^2}{|\tau|^2} \text{scal}_{g(t)}(\gamma(t))$$

for all $t \in [\tau, \tau/2]$. Thus, we conclude that

$$\text{scal}_{g(\tau/2)}(q) \geq \exp\left(-\frac{d_{g(t)}(p, q)^2}{|\tau|}\right) \text{scal}_{g(\tau)}(p).$$

Since $p, q \in M$ are arbitrary, the assertion follows. \hfill \Box

PROPOSITION 14
Suppose that the curvature tensor of $(M, g(t))$ lies in the cone $\tilde{C}$ for each $t \in (-\infty, 0)$. Moreover, suppose that

$$\limsup_{\tau \to -\infty} \frac{1}{\sqrt{|\tau|}} \text{diam}(M, g(\tau)) < \infty.$$ 

Then

$$\limsup_{\tau \to -\infty} \left[ |\tau| \sup_{M} \text{scal}_{g(\tau)} \right] < \infty.$$ 

Proof
Since the solution $g(t)$ is defined until time 0, we have

$$\inf_{M} \text{scal}_{g(\tau/2)} \leq \frac{n}{|\tau|}.$$
for each $\tau \in (-\infty, 0)$ (see, e.g., [5, Proposition 2.19]). Using Lemma 13, we deduce that
\[
\sup_M \text{scal}_{g(\tau)} \leq \frac{n}{|\tau|} \exp\left(\frac{\text{diam}(M, g(\tau))^2}{|\tau|}\right).
\]
From this, the assertion follows. \hfill \Box

Finally, we recall the following result due to B. Kostant (see [17, Corollary 2.2]).

**Proposition 15**

Let $(N, h)$ be a compact, simply connected Riemannian manifold of dimension $n \neq 5$ which is, topologically, a rational homology sphere. Then the holonomy representation of $(N, h)$ is complete; that is, $(N, h)$ has holonomy group $\text{SO}(n)$.

We now state the main result of this section.

**Theorem 16**

Let $g(t), t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact, even-dimensional manifold $M$. Suppose that the curvature tensor of $(M, g(t))$ lies in the interior of the cone $\hat{C}$ for each $t \in (-\infty, 0)$. Moreover, suppose that
\[
\limsup_{\tau \to -\infty} \frac{1}{\sqrt{|\tau|}} \text{diam}(M, g(\tau)) \leq \infty.
\]
Then $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

**Proof**

Suppose the assertion is false. By Theorem 11, we can find a sequence of points $(p_k, \tau_k) \in M \times (-\infty, 0)$ such that $\lim_{k \to \infty} \tau_k = -\infty$ and
\[
R_{g(\tau_k)} - \frac{1}{k} \text{scal}_{g(\tau_k)} I \notin \hat{C}
\]
at $p_k$. For each $k$, we consider the rescaled metrics
\[
\tilde{g}_k(t) = \frac{1}{|\tau_k|} g(|\tau_k| t), \quad t \in (-2, -1/2).
\]
For each $k$, the metrics $\tilde{g}_k(t), t \in (-2, -1/2)$, form a solution to the Ricci flow on $M$. By assumption, the diameter of $(M, \tilde{g}_k(t))$ has uniformly bounded diameter; moreover, it has uniformly bounded curvature by Proposition 14. Since $M$ is even-dimensional, we conclude that the injectivity radius of $(M, \tilde{g}_k(t))$ is uniformly bounded from below.

Hence, after passing to a subsequence if necessary, the sequence $(M, \tilde{g}_k(t))$ converges in the Cheeger-Gromov sense to some limiting solution $(M, \bar{g}(t))$ to the Ricci flow. This limiting solution is defined for all $t \in (-2, -1/2)$. Clearly, the curvature
tensor of \((M, \tilde{g}(t))\) lies in the cone \(\tilde{C}\) for each \(t \in (-2, -1/2)\). Moreover, it follows from (2) that the curvature tensor of \((M, \tilde{g}(-1))\) lies on the boundary of the cone \(\tilde{C}\) for some point \(q \in M\). By [6, Proposition 9], the manifold \((M, \tilde{g}(-1))\) has non-generic holonomy group; that is, \(\text{Hol}^0(M, \tilde{g}(-1)) \neq \text{SO}(n)\). On the other hand, it follows from the differentiable sphere theorem that the universal cover of \((M, \tilde{g}(-1))\) is diffeomorphic to \(S^n\) (see [7, Theorem 3]). By Proposition 15, the universal cover of \((M, \tilde{g}(-1))\) has holonomy group \(\text{SO}(n)\). This is a contradiction.

\[\square\]

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